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LINEAR COMBINATION OF DATA WITH LEAST ERROR OF DIFFERENCES

BY

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C O N T E N T S

INTRODUCTION

One of the many ways of obtaining from a set of observations a second smoothed or graduated set is to assume that the second set is a linear combination of the first. Thus if u denotes the column vector of n observed values, y that of the graduated and C the matrix performing the linear transformation, then $y = Cu$. This method was considered by W.F. Sheppard,[※] in the case where the observed data are equidistant, equally weighted and uncorrelated; the assumptions being that the sum of the squared coefficients in the transformation shall be a minimum; and that each y shall differ from a specified u by differences of u of order exceeding j, i.e. if the u's are already polynomial values of degree j, then the linear transformation leaves them unaltered. In this way each graduated value depends upon every observation, and not simply on those on either side as, for example, in the case of the centred finite summation formulae of Spencer or Woolhouse.[†] Sheppard points out that the solution of this problem yields precisely the same final results as that of fitting a curve of degree j to the u's by the method of least squares. A.C. Aitken[‡] has shown more recently how this problem in its two aspects may be solved much /

※ Journ. Inst. Act. Vol. 48, pp. 171-185.

† Whittaker and Robinson, Calculus of Observations pp. 286-290.

‡ Proc. Roy. Soc. Edin., Vol. 55, pp. 42-47.

much more concisely by using the matrix calculus, and indeed he gives the solution for the case where the u's are not subject to the above restricted conditions but may be of arbitrary functional type. The transformations which he derives for the restricted and general cases are

$$y = P(P'P)^{-1}P' \quad \text{(no correlation and equal weights)}$$

$$\text{and } y = P(P'V^{-1}P)^{-1}P'V^{-1} \text{ respectively,}$$

where P is a matrix of prescribed functional values by which the y's are expressed, and V is the symmetric variance matrix associated with the data u. $(V = [v_{ij}] = [\rho_{ij}\sigma_i\sigma_j])$

In Chapter I of this thesis the problem of graduation by linear combination is again considered, but with different minimal conditions. Firstly, what linear combination $y = Cu$ is such that the set of k^{th} differences $D_k y = D_k Cu$ has minimum sum of squared residuals and secondly, what linear combination $CD_k u$ of the k^{th} differences of the observed values produces a set of smoothed k^{th} differences with minimum sum squared residuals. Examples are given using both factorial polynomials and the orthogonal polynomials of Tchebychef. It is also shown that this problem leads to the same solution as that obtained by using Sheppard's original assumptions.

In Chapter II the linear combination C of observed data is considered where the y's are expressed in terms of the harmonic functions. The properties of the transforming matrix are established, and the Fourier coefficients are /

are given in matrix form. The question of estimate errors from residuals which is of prime importance in the examination of any physical phenomena associated with harmonic analysis, is also considered.

In the appendix tables of C are given for values of 2n, the number of data, equal to 4,6,8,10,12,16 and 24 with $k = 1,2,\ldots\ldots n$, the number of harmonics in the series. A bibliography of works consulted is also given.

I should like also to express here my sincere thanks to Dr. A.C. Aitken, F.R.S., for his constant guidance and encouragement throughout the work.

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CHAPTER I.

LINEAR COMBINATION OF DATA WITH LEAST ERROR OF DIFFERENCES

1. Let us consider n data

$$u = \{u_1 \; u_2 \; u_3 \ldots\ldots u_n\}$$

It is proposed to find what linear combination C of these data will produce a set of smoothed values

$$y = \{y_1 \; y_2 \; y_3 \ldots\ldots y_n\}$$

that is $y = Cu$ where

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & & c_{1n} \\ c_{21} & c_{22} & c_{23} & & c_{2n} \\ c_{31} & c_{32} & c_{33} & & c_{3n} \\ & & & & \\ c_{n1} & c_{n2} & c_{n3} & & c_{nn} \end{bmatrix}$$

subject to the following postulates:-

(l) that the u's are observations which are not correlated and which have the same standard error.

(II) that the mean square error, or sum square error, of the k^{th} differences of the smoothed y's is to be a minimum.

(III) that, if the u's are already values of a polynomial of the m^{th} degree $(m > k)$, the process of linear transformation leaves them unaltered, that is $Cu = u$.

(IV) that the moments of crude data and smoothed values up to the $(k-1)^{th}$ order shall agree. (To fix arbitrary constants).

Let /

Let D_k denote the differencing matrix

$$\begin{bmatrix} k_{(0)} & k_{(1)} & k_{(2)} & \cdots & k_{(k)} & \cdot & \cdot & \cdot \\ \cdot & k_{(0)} & k_{(1)} & \cdots & \cdots & k_{(k)} & \cdot & \cdot \\ \cdot & \cdot & k_{(0)} & \cdots & \cdots & \cdots & k_{(k)} & \cdot \\ & & & & & & & \ddots \\ \cdot & \cdot & \cdot & \cdots & k_{(0)} & k_{(1)} & \cdots & k_{(k)} \end{bmatrix}$$

of order $(n - k) \times n$.

where $k_{(r)}$ is the r^{th} term in the binomial expansion of $(-1+1)^k$.

Since $y = Cu$

$$D_k y = D_k C u$$

The sum squared error of the k^{th} differences of the smoothed values is the trace (sum of diagonal elements) of the matrix product

$$D_k C \, (D_k C)'$$
$$= D_k C C' D_k'.$$

Further let

$$u(x) = a_0 p_0(x) + a_1 p_1(x) + a_2 p_2(x) + \cdots\cdots + a_m p_m(x)$$

a polynomial of degree m in x,

so that

$$u = Pa$$

where u and a are column vectors and

$$P = \begin{bmatrix} p_0(1) & p_1(1) & p_2(1) & & p_m(1) \\ p_0(2) & p_1(2) & p_2(2) & & p_m(2) \\ p_0(3) & p_1(3) & p_2(3) & & p_m(3) \\ \\ p_0(n) & p_1(n) & p_2(n) & & p_m(n) \end{bmatrix}$$ of order $n \times (m+1)$

Postulate (III) above lays down that $CPa = Pa$ for all polynomials of degree m, that is for all vectors <u>a</u>

$$\therefore \; CP = P$$

Now T the trace of $D_k C (D_k C)'$ is the leading diagonal of /

of the product

$$\begin{bmatrix} \Delta^k c_{11} & \Delta^k c_{12} & \cdots & \Delta^k c_{1n} \\ \Delta^k c_{21} & \Delta^k c_{22} & \cdots & \Delta^k c_{2n} \\ \\ \Delta^k c_{n-k,1} & \Delta^k c_{n-k,2} & \cdots & \Delta^k c_{n-k,n} \end{bmatrix} \begin{bmatrix} \Delta^k c_{11} & \Delta^k c_{21} & \cdots & \Delta^k c_{n-k,1} \\ \Delta^k c_{12} & \Delta^k c_{22} & \cdots & \Delta^k c_{n-k,2} \\ \\ \Delta^k c_{1n} & \Delta^k c_{2n} & \cdots & \Delta^k c_{n-k,n} \end{bmatrix}$$

i.e. $T = \sum' (\Delta^k c_{1r})^2 + \sum' (\Delta^k c_{2r})^2 + \cdots + \sum' (\Delta^k c_{n-k,r})^2$, $\quad r=1$ to $r=n$.

$$\therefore \frac{1}{2}\frac{\partial T}{\partial c_{11}} = \Delta^k c_{11}$$

$$\frac{1}{2}\frac{\partial T}{\partial c_{21}} = \Delta^k c_{21} + k_{(1)} \Delta^k c_{11}$$

$$\frac{1}{2}\frac{\partial T}{\partial c_{31}} = \Delta^k c_{31} + k_{(1)} \Delta^k c_{21} + k_{(2)} \Delta^k c_{11}$$

$$\frac{1}{2}\frac{\partial T}{\partial c_{n-k,1}} = \Delta^k c_{n-k,1} + k_{(1)} \Delta^k c_{n-k-1,1} + \cdots + k_{(k)} \Delta^k c_{n-2k+1,1}$$

$$\therefore \frac{1}{2}\frac{\partial T}{\partial C} = D_k' D_k C \text{ ....} \quad \mathrm{I}$$

where C is or order (n-k,n).

Since this trace is to be a minimum subject to the condition that CP = P, i.e. $D_k CP = D_k P$

consider $(D_k CP - D_k P)\Lambda$ where Λ is the matrix of Lagrange undetermined multipliers.

$$\Lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} & \cdots & \lambda_{2n} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} & \cdots & \lambda_{3n} \\ \\ \lambda_{m+1,1} & \lambda_{m+1,2} & \lambda_{m+1,3} & \cdots & \lambda_{m+1,n} \end{bmatrix}$$

$$\therefore \ (D_k CP - D_k P)\Lambda$$

$$\left\{\begin{bmatrix} \Delta C_{11} & \Delta C_{12} & \cdots & \Delta C_{1n} \\ \Delta C_{21} & \Delta C_{22} & \cdots & \Delta C_{2n} \\ \Delta C_{31} & \Delta C_{32} & \cdots & \Delta C_{3n} \\ & & & \\ \Delta C_{n-k,1} & \Delta C_{n-k,2} & \cdots & \Delta C_{n-k,n} \end{bmatrix}\begin{bmatrix} p_0(1) & p_1(1) & p_2(1) & \cdots & p_m(1) \\ p_0(2) & p_1(2) & p_2(2) & \cdots & p_m(2) \\ p_0(3) & p_1(3) & p_2(3) & \cdots & p_m(3) \\ & & & & \\ p_0(n) & p_1(n) & p_2(n) & \cdots & p_m(n) \end{bmatrix}\right.$$

$$\left. - \begin{bmatrix} \Delta p_0(1) & \Delta p_1(1) & \Delta p_2(1) & \cdots & \Delta p_m(1) \\ \Delta p_0(2) & \Delta p_1(2) & \Delta p_2(2) & \cdots & \Delta p_m(2) \\ \Delta p_0(3) & \Delta p_1(3) & \Delta p_2(3) & \cdots & \Delta p_m(3) \\ & & & & \\ \Delta p_0(n-k) & \Delta p_1(n-k) & \Delta p_2(n-k) & \cdots & \Delta p_m(n-k) \end{bmatrix}\right\} \Lambda$$

$$\Delta^{K} = \Delta.$$

$$= \begin{bmatrix} \sum \Delta C_{1r} p_0(r) - \Delta p_0(1) & \sum \Delta C_{1r} p_1(r) - \Delta p_1(1) & \cdots & \sum \Delta C_{1r} p_m(r) - \Delta p_m(1) \\ \sum \Delta C_{2r} p_0(r) - \Delta p_0(2) & \sum \Delta C_{2r} p_1(r) - \Delta p_1(2) & \cdots & \sum \Delta C_{2r} p_m(r) - \Delta p_m(2) \\ \sum \Delta C_{3r} p_0(r) - \Delta p_0(3) & \sum \Delta C_{3r} p_1(r) - \Delta p_1(3) & \cdots & \sum \Delta C_{3r} p_m(r) - \Delta p_m(3) \\ & & & \\ \sum \Delta C_{n-k,r} p_0(r) - \Delta p_0(n-k) & \sum \Delta C_{n-k,r} p_1(r) - \Delta p_1(n-k) & \cdots & \sum \Delta C_{n-k,r} p_m(r) - \Delta p_m(n-k) \end{bmatrix}$$

$$\times \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} & \cdots & \lambda_{2n} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} & \cdots & \lambda_{3n} \\ & & & & \\ \lambda_{m+1,1} & \lambda_{m+1,2} & \lambda_{m+1,3} & \cdots & \lambda_{m+1,n} \end{bmatrix} \qquad r = 1 \text{ to } r = n.$$

Denote the leading diagonal of this matrix product by ϕ

then $\phi = \{\sum \Delta C_{1r} p_0(r) - \Delta p_0(1)\}\lambda_{11} + \{\sum \Delta C_{1r} p_1(r) - \Delta p_1(1)\}\lambda_{21} + \cdots + \{\sum \Delta C_{1r} p_m(r) - \Delta p_m(1)\}\lambda_{m+1,1}$

$+ \{\sum \Delta C_{2r} p_0(r) - \Delta p_0(2)\}\lambda_{12} + \{\sum \Delta C_{2r} p_1(r) - \Delta p_1(2)\}\lambda_{22} + \cdots + \{\sum \Delta C_{2r} p_m(r) - \Delta p_m(2)\}\lambda_{m+1,2}$

$+ \cdots\cdots$

$$+\left\{\sum \Delta C_{n-k,r}\, p_0(r) - \Delta p_0(n-k)\right\}\lambda_{1n} + \cdots\cdots + \left\{\sum \Delta C_{n-k,r}\, p_m(r) - \Delta p_m(n-k)\right\}\lambda_{m+1,n}$$

$$\therefore \frac{\partial \phi}{\partial C_{11}} = p_0(1)\lambda_{11} + p_1(1)\lambda_{21} + \cdots\cdots + p_m(1)\lambda_{m+1,1}$$

$$= \sum p_r(1)\lambda_{r+1,1} \qquad \left\{\begin{array}{l} r = 0 \text{ to} \\ r = m \end{array}\right.$$

$$\frac{\partial \phi}{\partial C_{21}} = \sum p_r(1)\lambda_{r+1,2} + k_{(1)}\sum p_r(1)\lambda_{r+1,1}$$

$$\frac{\partial \phi}{\partial C_{31}} = \sum p_r(1)\lambda_{r+1,3} + k_{(1)}\sum p_r(1)\lambda_{r+1,2} + k_{(2)}\sum p_r(1)\lambda_{r+1,1}$$

$$\frac{\partial \phi}{\partial C_{n-k,1}} = \sum p_r(1)\lambda_{r+1,n-k} + k_{(1)}\sum p_r(1)\lambda_{r+1,n-k-1} + \cdots\cdots + k_{(n-k-1)}\sum p_r(1)\lambda_{r+1,1}$$

$$\therefore \begin{bmatrix} \frac{\partial \phi}{\partial C_{11}} \\ \frac{\partial \phi}{\partial C_{21}} \\ \frac{\partial \phi}{\partial C_{31}} \\ \vdots \\ \frac{\partial \phi}{\partial C_{n-k,1}} \end{bmatrix} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ k_{(1)} & 1 & \cdot & \cdot \\ k_{(2)} & k_{(1)} & 1 & \cdot \\ & & & \\ k_{(n-k-1)} & k_{(n-k-2)} & & 1 \end{bmatrix} \begin{bmatrix} \lambda_{11} & \lambda_{21} & \lambda_{31} & \cdots\cdots \lambda_{m+1,1} \\ \lambda_{12} & \lambda_{22} & \lambda_{32} & \cdots\cdots \lambda_{m+1,2} \\ & & & \\ \lambda_{1,n} & \lambda_{2,n} & \lambda_{3n} & \cdots\cdots \lambda_{m+1,n} \end{bmatrix} \begin{bmatrix} p_0(1) \\ p_1(1) \\ \\ p_m(1) \end{bmatrix}$$

$$\therefore \frac{\partial \phi}{\partial C} = D'\Lambda' P' \cdots\cdots\cdots\cdots \text{ II.}$$

From I and II we have

$$D'DC = D'\Lambda' P'$$

subject to the condition $CP = P$

$$\therefore DC = \Lambda' P'$$
$$\therefore DCP = \Lambda' P'P$$
$$\therefore DP = \Lambda' P'P \qquad \text{since } CP = P$$
$$\therefore DP(P'P)^{-1} = \Lambda'$$
$$\therefore DP(P'P)^{-1}P' = \Lambda' P'$$
$$= DC$$
$$\therefore \underline{D_k C = D_k P(P'P)^{-1}P'} \cdots\cdots\cdots\cdots \text{ III.}$$

The result is very similar to that obtained by using W.F. Sheppard's postulate that the graduated values themselves, and not their k^{th} differences, have minimum sum squared error. A.C. Aitken's matrix solution is $C = P(P'P)^{-1}P'$. We cannot, of course, cancel out D_k from III since D_k is rectangular.

2. We shall now show that

$$C = P(P'P)^{-1}P' \quad \ldots\ldots\ldots\ldots\ldots \quad \text{I}$$

$$\text{and } D_k C = D_k P(P'P)^{-1}P' \quad \ldots\ldots\ldots\ldots \quad \text{II}$$

lead to an identical solution.

The graduated values y which have been obtained by operating on the observed data u directly by C, or which have been obtained from the differences $D_k Cu$, have the same k^{th} differences, hence the fitted values y from the two methods can differ from each other only to the extent of a polynomial of degree k -1; this follows because the k^{th} differences of a polynomial of degree k -1 are zero.

Let us fit our crude data <u>u</u> by means of Tchebychef polynomials, so that

$$u = a_0 t_0(x) + a_1 t_1(x) + \text{-------} + a_{n-1} t_{n-1}(x)$$

a polynomial of degree n -1.

Let $u_{(1}$ denote the terms $a_0 t_0(x) + a_1 t_1(x) + \cdots + a_{k-1} t_{k-1}(x)$

$u_{(2}$ " " " $a_k t_k(x) + \cdots\cdots + a_m t_m(x)$

$u_{(3}$ " " " $a_{m+1} t_{m+1}(x) + \cdots\cdots + a_{n-1} t_{n-1}(x)$

then $u = u_{(1} + u_{(2} + u_{(3}$

It has been shown that

$$y = Cu = C\left[u_{(1} + u_{(2} + u_{(3}\right] = u_{(1} + u_{(2} \quad \text{--------} \quad \text{I.}$$

that /

that is ordinary least square fitting of a polynomial of degree m preserves the first (m + 1) terms of the whole Tchebychef expansion and obliterates the rest.

Let us further suppose that the fitted values y which have been derived from the k^{th} difference postulate are

$$y = v_{(1} + v_{(2} \quad \ldots\ldots\ldots \text{ II.}$$

that is the values of y in I and II differ only from each other to the extent of a polynomial of degree k -1. But postulate IV. lays down that the moments of the crude data and of the smoothed values up to the $(k-1)^{th}$ order shall agree i.e. the moments of I and II up to the $(k-1)^{th}$ agree.

Moments of $u_{(1}$ and $v_{(1}$ for arbitrary $u_{(1}$ must agree

i.e. $u_{(1} = v_{(1}$

and we therefore have an identical solution.

3. Assuming that

$$C = P(P'P)^{-1}P'$$

and $D_k C = D_k P(P'P)^{-1}P'$

lead to an identical solution, properties of C could be established. For example, if we consider first differences

DCu yields

$$\begin{aligned} \Delta y_1 &= \textstyle\sum_1 (C_{2r} - C_{1r}) u_r \\ \Delta y_2 &= \textstyle\sum_1 (C_{3r} - C_{2r}) u_r \\ &\vdots \\ \Delta y_{n-1} &= \textstyle\sum_1 (C_{nr} - C_{n-1,r}) u_r \end{aligned} \qquad \begin{aligned} r &= 1 \\ \text{to } r &= n \end{aligned}$$

Since the moments of order zero of the graduated and ungraduated are equal, and if <u>x</u> is the smoothed value of u, obtained from these first differences then

$$nx + (n-1)\Delta y_1 + (n-2)\Delta y_2 + \cdots \; \Delta y_{n-1} = u_1 + u_2 + \cdots + u_n$$

$$\therefore \; nx + (n-1)\textstyle\sum_1 (C_{2r} - C_{1r})u_r + (n-2)\sum_1 (C_{3r} - C_{2r})u_r + \cdots + \sum_1 (C_{nr} - C_{n-1,r})u_r = \sum u_r$$

Since /

Since the solutions are identical, this smoothed value x must be the same as that obtained directly from the linear transformation C

i.e. $$x = \sum_{r=1}^{n}{}' C_{1r} u_r$$

whence
$$n\sum_1' C_{1r} u_r + (n-1)\sum_1' (C_{2r} - C_{1r}) u_r + \cdots\cdots + \sum' (C_{nr} - C_{n-1,r}) u_r = \sum_1' u_r.$$

This must be true for any arbitrary values u, whence
$$n\, C_{1r} + (n-1)(C_{2r} - C_{1r}) + (n-2)(C_{3r} - C_{2r}) + \cdots + (C_{nr} - C_{n-1,r}) = 1$$
$$\therefore\ C_{1r} + C_{2r} + C_{3r} + \cdots\cdots + C_{nr} = 1$$
for all values of r from 1 to n, that is the sum of the elements of any row of C is unity.

Moment Properties of C.

Consider the constant set $\{1, 1, 1, \ldots\ 1\}$

If these are graduated by C they must be reproduced
$$\therefore\ C\{1,1,1,\ldots\ 1\} = \{1,1,1,\ldots\ 1\}$$
$$\therefore\ C_{1r} + C_{2r} + C_{3r} + \cdots\cdots + C_{nr} = 1$$
for values of r from 1 to n.

Since moments of order 1 are reproduced
$$C\{1, 2, 3, \ldots\ n\} = \{1, 2, 3, \ldots\ n\}$$
$$\therefore\ C_{1r} + 2C_{2r} + 3C_{3r} + \cdots + n\,C_{nr} = r$$

In the same way moments of order $j \leq m$, the degree of the polynomial, are reproduced whence
$$C_{1r} + 2^j C_{2r} + 3^j C_{3r} + \cdots + n^j C_{nr} = r^j.$$

For example if

$$C = \frac{1}{168}\begin{bmatrix} 119 & 63 & 21 & -7 & -21 & -21 & -7 & 21 \\ 63 & 47 & 33 & 21 & 11 & 3 & -3 & -7 \\ 21 & 33 & 39 & 39 & 33 & 21 & 3 & -21 \\ -7 & 21 & 39 & 47 & 45 & 33 & 11 & -21 \\ -21 & 11 & 33 & 45 & 47 & 39 & 21 & -7 \\ -21 & 3 & 21 & 33 & 39 & 39 & 33 & 21 \\ -7 & -3 & 3 & 11 & 21 & 33 & 47 & 63 \\ 21 & -7 & -21 & -21 & -7 & 21 & 63 & 119 \end{bmatrix}$$

Then the sum of the elements of any row is unity.

Also

$$\frac{1}{168}\left[63 + 2(47) + 3(33) + 4(21) + 5(11) + 6(3) + 7(-3) + 8(-7)\right]$$

$$= \frac{1}{168}\left[63 + 94 + 99 + 84 + 55 + 18 - 21 - 56\right]$$

$$= \frac{1}{168}\left[336\right]$$

$$= \underline{\underline{2}}$$

and

$$\frac{1}{168}\left[63 + 2^2(47) + 3^2(33) + 4^2(21) + 5^2(11) + 6^2(3) + 7^2(-3) + 8^2(-7)\right]$$

$$= \frac{1}{168}\left[63 + 188 + 297 + 336 + 275 + 108 - 147 - 448\right]$$

$$= \frac{1}{168}\left[672\right]$$

$$= \underline{\underline{2^2}}$$

In this case higher moments cannot be considered as C is derived from a polynomial of the second degree.

4. The only difficulty which may arise in evaluating C is the finding of the reciprocal of P′P, particularly when the order of this matrix is high. In this paragraph we shall show the advantage of using Tchebychef polynomials, and incidentally show again that the sum of the elements of any row of C is unity.

If $\quad y(x) = a_0 t_0(x) + a_1 t_1(x) + \cdots\cdots + a_r t_r(x)$

then

$$P'P = \begin{bmatrix} t_0(0) & t_0(1) & t_0(2) & & t_0(n-1) \\ t_1(0) & t_1(1) & t_1(2) & & t_1(n-1) \\ t_2(0) & t_2(1) & t_2(2) & & t_2(n-1) \\ & & & & \\ & & & & \\ t_r(0) & t_r(1) & t_r(2) & & t_r(n-1) \end{bmatrix} \begin{bmatrix} t_0(0) & t_1(0) & t_2(0) & & t_r(0) \\ t_0(1) & t_1(1) & t_2(1) & & t_r(1) \\ t_0(2) & t_1(2) & t_2(2) & & t_r(2) \\ & & & & \\ & & & & \\ t_0(n-1) & t_1(n-1) & t_2(n-1) & & t_r(n-1) \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{x=0}^{n-1} t_0(x)^2 & & & \\ & \sum_{x=0}^{n-1} t_1(x)^2 & & \\ & & \ddots & \\ & & & \sum_{x=0}^{n-1} t_r(x)^2 \end{bmatrix}$$

all the non diagonal elements being zero since

$$\sum_{x=0}^{n-1} t_r(x)\, t_s(x) = 0 \quad , \quad r \neq s.$$

$$\therefore (P'P)^{-1} = \begin{bmatrix} \frac{1}{\sum t_0(x)^2} & & & \\ & \frac{1}{\sum t_1(x)^2} & & \\ & & \ddots & \\ & & & \frac{1}{\sum t_r(x)^2} \end{bmatrix}$$

$$\therefore (P'P)^{-1}P' = \begin{bmatrix} \frac{1}{\sum t_0(x)^2} & & & \\ & \frac{1}{\sum t_1(x)^2} & & \\ & & \ddots & \\ & & & \frac{1}{\sum t_r(x)^2} \end{bmatrix} \begin{bmatrix} t_0(0) & t_0(1) & t_0(2) & t_0(n-1) \\ t_1(0) & t_1(1) & t_1(2) & t_1(n-1) \\ \vdots & & & \\ t_r(0) & t_r(1) & t_r(2) & t_r(n-1) \end{bmatrix}$$

$$\therefore P(P'P)^{-1}P' =$$

$$\begin{bmatrix} t_0(0) & t_1(0) & t_r(0) \\ t_0(1) & t_1(1) & t_r(1) \\ \vdots & & \\ t_0(n-1) & t_1(n-1) & t_r(n-1) \end{bmatrix} \begin{bmatrix} \frac{t_0(0)}{\sum t_0(x)^2} & \frac{t_0(1)}{\sum t_0(x)^2} & \frac{t_0(n-1)}{\sum t_0(x)^2} \\ \frac{t_1(0)}{\sum t_1(x)^2} & \frac{t_1(1)}{\sum t_1(x)^2} & \frac{t_1(n-1)}{\sum t_1(x)^2} \\ \vdots & & \\ \frac{t_r(0)}{\sum t_r(x)^2} & \frac{t_r(1)}{\sum t_r(x)^2} & \frac{t_r(n-1)}{\sum t_r(x)^2} \end{bmatrix}$$

$\therefore$

Sum of the elements of the $(p + 1)^{th}$ row of $P(P'P)^{-1}P'$ is obtained by premultiplying each column of $(P'P)^{-1}P'$ by the $(p + 1)^{th}$ row of P and summing; this gives

$$\frac{t_0(p)}{\sum t_0(x)^2}\left[t_0(0) + t_0(1) + \cdots\cdots + t_0(n-1)\right]$$

$$+ \frac{t_1(p)}{\sum t_1(x)^2}\left[t_1(0) + t_1(1) + \cdots\cdots + t_1(n-1)\right]$$

$$\vdots$$

$$+ \frac{t_r(p)}{\sum t_r(x)^2}\left[t_r(0) + t_r(1) + \cdots\cdots + t_r(n-1)\right]$$

But $\sum_{x=0}^{n-1} t_r(x) = 0 \quad , r \neq 0$ ※

and $\sum_{x=0}^{n-1} t_0(x) = n, \quad t_0(r) = 1, \quad \sum_{x=0}^{n-1} t_0(x)^2 = n$

$\therefore$ Sum of the elements of any row is unity.

※ G.J. Lidstone, Journ. Inst. Act. Vol. 64 Pt. II. p. 145.

EXAMPLE /

EXAMPLE.

Let us graduate the 8 data 1,2,5,8,10,15,20,30, assuming that our representation is a polynomial function of the second degree. In this example we shall use factorial polynomials which are commonest in the calculus of finite differences.

The general polynomial is

$$y = a_0 + a_1 x_{(1)} + a_2 x_{(2)} + \cdots\cdots + a_r x_{(r)}$$

$$\text{where} \quad x_r = \frac{x(x-1)(x-2)\cdots\cdots(x-r+1)}{r!}$$

In this example

$$P'P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \cdot & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \cdot & \cdot & 1 & 3 & 6 & 10 & 15 & 21 \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot \\ 1 & 1 & \cdot \\ 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 4 & 6 \\ 1 & 5 & 10 \\ 1 & 6 & 15 \\ 1 & 7 & 21 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 28 & 56 \\ 28 & 140 & 322 \\ 56 & 322 & 812 \end{bmatrix}$$

$$\therefore (P'P)^{-1} = \frac{1}{3528} \begin{bmatrix} 2499 & -1176 & 294 \\ -1176 & 840 & -252 \\ 294 & -252 & 84 \end{bmatrix} = \frac{1}{168} \begin{bmatrix} 119 & -56 & 14 \\ -56 & 40 & -12 \\ 14 & -12 & 4 \end{bmatrix}$$

$$\therefore P(P'P)^{-1} = \frac{1}{168} \begin{bmatrix} 1 & \cdot & \cdot \\ 1 & 1 & \cdot \\ 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 4 & 6 \\ 1 & 5 & 10 \\ 1 & 6 & 15 \\ 1 & 7 & 21 \end{bmatrix} \begin{bmatrix} 119 & -56 & 14 \\ -56 & 40 & -12 \\ 14 & -12 & 4 \end{bmatrix} \quad \frac{1}{168} \begin{bmatrix} 119 & -56 & 14 \\ 63 & -16 & 2 \\ 21 & 12 & -6 \\ -7 & 28 & -10 \\ -21 & 32 & -10 \\ -21 & 24 & -6 \\ -7 & 4 & 2 \\ 21 & -28 & 4 \end{bmatrix}$$

$\therefore P(P'P)^{-1}P'$

$$= \frac{1}{168}\begin{bmatrix} 119 & 63 & 21 & -7 & -21 & -21 & -7 & 21 \\ 63 & 47 & 33 & 21 & 11 & 3 & -3 & -7 \\ 21 & 33 & 39 & 39 & 33 & 21 & 3 & -21 \\ -7 & 21 & 39 & 47 & 45 & 33 & 11 & -21 \\ -21 & 11 & 33 & 45 & 47 & 39 & 21 & -7 \\ -21 & 3 & 21 & 33 & 39 & 39 & 33 & 21 \\ -7 & -3 & 3 & 11 & 21 & 33 & 47 & 63 \\ 21 & -7 & -21 & -21 & -7 & 21 & 63 & 119 \end{bmatrix}$$

(Check: sum of elements in each row and column unity).

Performing this transformation of the given data we get

$$\frac{259}{168},\ \frac{375}{168},\ \frac{669}{168},\ \frac{1141}{168},\ \frac{1791}{168},\ \frac{2619}{168},\ \frac{3625}{168},\ \frac{4809}{168}$$

(Check: Sum of graduated data $\frac{15288}{168} = 91 =$ Sum of given data)

The smoothed first differences are

$$\frac{116}{168},\ \frac{294}{168},\ \frac{472}{168},\ \frac{650}{168},\ \frac{828}{168},\ \frac{1006}{168},\ \frac{1184}{168}$$

The second differences, $\frac{178}{168}$, are constant since the assumption is that the smoothed values lie on a curve of the second degree.

Further,

$$DC = \frac{1}{168}\begin{bmatrix} -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 \end{bmatrix}\begin{bmatrix} 119 & 63 & 21 & -7 & -21 & -21 & -7 & 21 \\ 63 & 47 & 33 & 21 & 11 & 3 & -3 & -7 \\ 21 & 33 & 39 & 39 & 33 & 21 & 3 & -21 \\ -7 & 21 & 39 & 47 & 45 & 33 & 11 & -21 \\ -21 & 11 & 33 & 45 & 47 & 39 & 21 & -7 \\ -21 & 3 & 21 & 33 & 39 & 39 & 33 & 21 \\ -7 & -3 & 3 & 11 & 21 & 33 & 47 & 63 \\ 21 & -7 & -21 & -21 & -7 & 21 & 63 & 119 \end{bmatrix}$$

$$= \frac{1}{168}\begin{bmatrix} -56 & -16 & 12 & 28 & 32 & 24 & 4 & -28 \\ -42 & -14 & 6 & 18 & 22 & 18 & 6 & -14 \\ -28 & 12 & \cdot & 8 & 12 & 12 & 8 & \cdot \\ -14 & -10 & -6 & -2 & 2 & 6 & 10 & 14 \\ \cdot & -8 & -12 & -12 & -8 & \cdot & 12 & 28 \\ 14 & -6 & -18 & -22 & -18 & -6 & 14 & 42 \\ 28 & -4 & -24 & -32 & -28 & -12 & 16 & 56 \end{bmatrix}$$

Operating this matrix on the data above we get

$\frac{1}{168}$ (116, 294, 472, 650, 828, 1006, 1184).

If we assume that the sum of the u's is equal to the sum of the smoothed y's we obtain at once the smoothed data

$\frac{1}{168}$ (259, 375, 669, 1141, 1791, 2619, 3625, 4809)

as before.

5. Let us now consider the same problem as above with this difference, that we replace postulate (II) by the following postulate:-

"The linear combination of the k^{th} differences of the <u>observed values</u>, that is, $CD_K u_i$ is to have minimum sum squared error."

Proceeding in a manner somewhat similar to the former we now have to minimise the trace of the matrix product

$$(CD_k)(CD_k)' \quad \text{i.e.} \quad CD_K D_K' C'$$

subject to the condition that CP = P

$$CD_K D_K' C' =$$

$$C D_K D_K' C' =$$

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & & c_{1n} \\ c_{21} & c_{22} & c_{23} & & c_{2n} \\ c_{31} & c_{32} & c_{33} & & c_{3n} \\ \\ c_{n1} & c_{n2} & c_{n3} & & c_{nn} \end{bmatrix} \begin{bmatrix} k_{(0)} & k_{(1)} & k_{(2)} & & \\ \cdot & k_{(0)} & k_{(1)} & & \\ \cdot & \cdot & k_{(0)} & & \\ & & & \ddots & \\ \cdot & \cdot & \cdot & & k_{(0)} \end{bmatrix} \begin{bmatrix} k_{(0)} & \cdot & & \\ k_{(1)} & k_{(0)} & \cdot & \\ k_{(2)} & k_{(1)} & k_{(0)} & \cdot \\ \\ & & & \end{bmatrix} \begin{bmatrix} c_{11} & c_{21} & c_{31} & c_{n1} \\ c_{12} & c_{22} & c_{32} & c_{n2} \\ c_{13} & c_{23} & c_{33} & c_{n3} \\ \\ c_{1n} & c_{2n} & c_{3n} & c_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} k_{(0)} c_{11} & k_{(1)} c_{11} + k_{(0)} c_{12} & \cdots\cdots & k_{(K)} c_{1,n-k} + k_{(K-1)} c_{1,n-k-1} + \cdots + k_{(0)} c_{1n} \\ k_{(0)} c_{21} & k_{(1)} c_{21} + k_{(0)} c_{22} & \cdots\cdots & k_{(K)} c_{2,n-k} + k_{(K-1)} c_{2,n-k-1} + \cdots + k_{(0)} c_{2n} \\ k_{(0)} c_{31} & k_{(1)} c_{31} + k_{(0)} c_{32} & \cdots\cdots & k_{(K)} c_{3,n-k} + k_{(K-1)} c_{3,n-k-1} + \cdots + k_{(0)} c_{3n} \\ \\ k_{(0)} c_{n1} & k_{(1)} c_{n1} + k_{(0)} c_{n2} & \cdots\cdots & k_{(K)} c_{n,n-k} + k_{(K-1)} c_{n,n-k-1} + \cdots + k_{(0)} c_{nn} \end{bmatrix}$$

$$\times \begin{bmatrix} k_{(0)} c_{11} & k_{(0)} c_{21} & \cdots\cdots & k_{(0)} c_{n1} \\ k_{(1)} c_{11} + k_{(0)} c_{12} & k_{(1)} c_{21} + k_{(0)} c_{22} & \cdots\cdots & k_{(1)} c_{n1} + k_{(0)} c_{n2} \\ k_{(2)} c_{11} + k_{(1)} c_{12} + k_{(0)} c_{13} & k_{(2)} c_{21} + k_{(1)} c_{22} + k_{(0)} c_{23} & \cdots\cdots & k_{(2)} c_{n1} + k_{(1)} c_{n2} + k_{(0)} c_{n3} \\ \\ k_{(K)} c_{1,n-k} + \cdots\cdots + k_{(0)} c_{1n} & k_{(K)} c_{2,n-k} + \cdots + k_{(0)} c_{2n} & \cdots\cdots & k_{(K)} c_{n,n-k} + \cdots + k_{(0)} c_{n,n} \end{bmatrix}$$

$$\therefore T = \sum \left\{ \left[k_{(0)} c_{r1} \right]^2 + \left[k_{(1)} c_{r1} + k_{(0)} c_{r2} \right]^2 + \cdots\cdots + \left[k_{(k)} c_{r,n-k} + \cdots\cdots + k_{(0)} c_{r,n} \right]^2 \right\}$$

$$\therefore \frac{1}{2} \frac{\partial T}{\partial c_{r1}} = \left[k_{(0)} \right]^2 c_{r1} + k_{(1)} \left[k_{(1)} c_{11} + k_{(0)} c_{r2} \right] + \cdots + k_{(K)} \left[k_{(K)} c_{r1} + \cdots\cdots + k_0 c_{r,k+1} \right]$$

$$= \begin{bmatrix} c_{r1} & c_{r2} & \cdots\cdots\cdots & c_{r,k+1} \end{bmatrix} \begin{bmatrix} k_{(0)}^2 + k_{(1)}^2 + \cdots\cdots + k_{(k)}^2 \\ k_{(0)} k_{(1)} + k_{(1)} k_{(2)} + \cdots + k_{(k-1)} k_{(K)} \\ k_{(0)} k_{(2)} + \cdots\cdots\cdots + k_{(k-2)} k_{(K)} \\ k_{(0)} k_{(3)} + \cdots\cdots + k_{(k-3)} k_{(k)} \\ \\ k_{(0)} k_{(K)} \end{bmatrix}$$

whence $\quad \dfrac{1}{2} \dfrac{\partial T}{\partial C} = C D_K D_K'$

Introducing /

Introducing Lagrange multipliers as before we have

$$C\,DD' = \Lambda' P'$$

$$CP = P$$

$$C = \Lambda' P' (DD')^{-1}$$

$$CP = \Lambda' P' (DD')^{-1} P$$

$$P = \Lambda' P' (DD')^{-1} P$$

$$P[P'(DD')^{-1}P]^{-1} = \Lambda'$$

$$P[P'(DD')^{-1}P]^{-1}P' = \Lambda' P'$$

$$= CDD'$$

$$\therefore \underline{C = P[P'(DD')^{-1}P]^{-1}P'(DD')^{-1}}$$

6. Since the differences of the u's are necessarily functions of the u's which follow, these differences are correlated. The variance matrix V in this case will be $D_k D_k'$. Since the u's are polynomials of degree m, their k^{th} differences will be of degree $(m-k)$. Let the smoothed k^{th} differences $D_k y$ be represented by the polynomial Pa. The sum of squared residuals is given by

$$S^2 = (D_k u - Pa)'(DD')^{-1}(D_k u - Pa) \quad *$$

$\frac{\partial S^2}{\partial a}$ yields $\quad P'(DD')^{-1}Pa = P'(DD')^{-1}Du$

$$\therefore a = [P'(DD')^{-1}P]^{-1}P'(DD')^{-1}$$

$$\therefore \underline{Dy = P[P'(DD')^{-1}P]^{-1}P'(DD')^{-1}Du}$$

7. We shall now consider in matrix form the k^{th} differences of the y's as a linear combination of the k^{th} differences of the u's, the graduated y's having first been derived from the original postulate that they themselves /

* A.C. Aitken, Proc. Roy. Soc. Edin. Vol. 55 p. 45.

themselves have minimum sum squared error, that is

$$y = Cu \,, \qquad C = P(P'P)^{-1}P'$$

The case for first differences is here considered, but the same method may be applied to differences of a higher order.

$$\Delta y_1 = y_2 - y_1 = \sum_{r=1}^{n} (C_{2r} - C_{1r}) u_r$$
$$\Delta y_2 = y_3 - y_2 = \sum_{r=1}^{n} (C_{3r} - C_{2r}) u_r$$
$$\Delta y_{n-1} = y_n - y_{n-1} = \sum_{r=1}^{n} (C_{nr} - C_{n-1,r}) u_r$$

This expresses the difference of the y's as a linear combination of the u's themselves.

Now $\Delta y_1 = (C_{21} - C_{11}) u_1 + (C_{22} - C_{12}) u_2 + \cdots\cdots + (C_{2n} - C_{1n}) u_n$

$$= (C_{21} - C_{11})(u_1 - u_2) + (C_{21} - C_{11} + C_{22} - C_{12})(u_2 - u_3) + \cdots$$
$$+ (C_{21} - C_{11} + C_{22} - C_{12} + \cdots + C_{2,n-1} - C_{1,n-1})(u_{n-1} - u_n)$$
$$+ \left(\sum_{r=1}^{n} C_{2r} - C_{1r}\right) u_n$$

The last term vanishes since $\sum_{r=1}^{n} C_{2r} = \sum_{r=1}^{n} C_{1r} = 1$

$$\therefore \Delta y_1 = -\left[\Delta C_{11},\ \Delta C_{11} + \Delta C_{12},\ \ldots\ \Delta C_{11} + \Delta C_{12} + \cdots + \Delta C_{1,n-1}\right]\left\{\Delta u_1,\ \Delta u_2,\ \ldots\ \Delta u_{n-1}\right\}$$

and similarly

$$\Delta y_r = -\left[\Delta C_{r1},\ \Delta C_{r1} + \Delta C_{r2},\ \ldots\ \Delta C_{r1} + \Delta C_{r2} + \cdots + \Delta C_{r,n-1}\right]\left\{\Delta u_1,\ \Delta u_2,\ \ldots\ \Delta u_{n-1}\right\}$$

$$Dy = -\begin{bmatrix} \Delta C_{11} & \Delta C_{11} + \Delta C_{12} & \cdots & \Delta C_{11} + \Delta C_{12} + \cdots + \Delta C_{1,n-1} \\ \Delta C_{21} & \Delta C_{21} + \Delta C_{22} & \cdots & \Delta C_{21} + \Delta C_{22} + \cdots + \Delta C_{2,n-1} \\ & & & \\ \Delta C_{n-1,1} & \Delta C_{n-1,1} + \Delta C_{n-1,2} & \cdots & \Delta C_{n-1,1} + \Delta C_{n-1,2} + \cdots + \Delta C_{n-1,n-1} \end{bmatrix} \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \\ \\ \Delta u_{n-1} \end{bmatrix}$$

$$= -\begin{bmatrix} \Delta C_{11} & \Delta C_{12} & \Delta C_{1,n-1} \\ \Delta C_{21} & \Delta C_{22} & \Delta C_{2,n-1} \\ \Delta C_{31} & \Delta C_{32} & \Delta C_{3,n-1} \\ & & \\ \Delta C_{n-1,1} & \Delta C_{n-1,2} & \Delta C_{n-1,n-1} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & & 1 \\ \cdot & 1 & 1 & 1 & & 1 \\ \cdot & \cdot & 1 & 1 & & 1 \\ & & & & & \\ \cdot & \cdot & \cdot & \cdot & & 1 \end{bmatrix} \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \\ \Delta u_3 \\ \\ \Delta u_{n-1} \end{bmatrix}$$

But

$$\begin{bmatrix} 1 & 1 & 1 & 1 & & 1 & 1 \\ \cdot & 1 & 1 & 1 & & 1 & 1 \\ \cdot & \cdot & 1 & 1 & & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & 1 \end{bmatrix} = -(D')^{-1}$$

$$\therefore \; Dy = DC(D')^{-1}$$

$$\text{i.e.} \; \underline{Dy = DP(P'P)^{-1}P'(D')^{-1}Du}$$

8. In evaluating this Linear Transformation the function $D_k D_k'$ occurs, and the following lemma is now established.

The elements of the matrix $D_k D_k'$ are the binomial coefficients of the expansion $(1 - t)^{2k}$

Let $\quad (1-t)^k = c_0 + c_1 t + c_2 t^2 + \cdots + c_k t^k$

$$\begin{bmatrix} c_0 & c_1 & c_2 & & c_k & \cdot & \cdot & \cdot \\ \cdot & c_0 & c_1 & & & c_k & \cdot & \cdot \\ \cdot & \cdot & c_0 & & & & c_k & \cdot \\ & & & & & & \ddots & \\ \cdot & \cdot & \cdot & & & & & c_k \end{bmatrix} \begin{bmatrix} c_0 & \cdot & & & \cdot \\ c_1 & c_0 & & & \cdot \\ c_k & & & & \\ & c_k & & & \\ & & & \ddots & c_k \end{bmatrix}$$

$$= \begin{bmatrix} \sum_0^k c_r^2 & \sum_1^k c_r c_{r-1} & \sum_2^k c_r c_{r-2} & \cdots \\ \sum_1^k c_r c_{r-1} & \sum_0^k c_r^2 & \sum_1^k c_r c_{r-1} & \cdots \\ \sum_2^k c_r c_{r-2} & \sum_1^k c_r c_{r-1} & \sum_0^k c_r^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Now $(1-t)^k \left(1 - \frac{1}{t}\right)^k = (c_0 + c_1 t + \cdots + c_k t^k)\left(c_0 + \frac{c_1}{t} + \cdots + \frac{c_k}{t^k}\right)$

$$\therefore (-)^k \frac{1}{t^k}(1-t)^{2k} = \frac{c_0 c_k}{t^k} + \sum_{k-1}^{k} c_r c_{r-(k-1)} \frac{1}{t^{k-1}} + \sum_{k-2}^{k} c_r c_{r-(k-2)} \frac{1}{t^{k-2}} + \cdots$$
$$+ \sum_1^k c_r c_{r-1} \frac{1}{t} + \sum_0^k c_r^2 + \sum_1^k c_r c_{r-1}\, t + \cdots + c_0 c_k t^k$$

whence /

whence it follows at once that the coefficients in $(1-t)^{2k}$ are the elements of $D_K D_K'$.

EXAMPLE.

To obtain the smoothed first differences of the data 1,2,5,8,10,15,20,30 by performing the transformations

$$P[P'(DD')^{-1}P]^{-1}P'(DD')^{-1}$$

and $DP(P'P)^{-1}P'(D')^{-1}$

on the first differences of the observed data, namely,

1, 3, 3, 2, 5, 5, 10.

For first differences the value of $(DD')^{-1}$ is

$$\begin{bmatrix} n & n-1 & n-2 & \cdots & 3 & 2 & 1 \\ n-1 & 2(n-1) & 2(n-2) & \cdots & 6 & 4 & 2 \\ n-2 & 2(n-2) & 3(n-2) & \cdots & 9 & 6 & 3 \\ & & & & & & \\ 3 & 6 & 9 & \cdots & 3(n-2) & 2(n-2) & n-2 \\ 2 & 4 & 6 & \cdots & 2(n-2) & 2(n-1) & n-1 \\ 1 & 2 & 3 & \cdots & n-2 & n-1 & n \end{bmatrix}$$

In this example the value of $P'(DD')^{-1}P$ is

$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 6 & 12 & 10 & 8 & 6 & 4 & 2 \\ 5 & 10 & 15 & 12 & 9 & 6 & 3 \\ 4 & 8 & 12 & 16 & 12 & 8 & 4 \\ 3 & 6 & 9 & 12 & 15 & 10 & 5 \\ 2 & 4 & 6 & 8 & 10 & 12 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{bmatrix}\begin{bmatrix} 1 & . \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{bmatrix}$$

$$= 42\begin{bmatrix} 1 & 3 \\ 3 & 10 \end{bmatrix}$$

$$\therefore\ [P'(DD')^{-1}P]^{-1} = \frac{1}{42}\begin{bmatrix} 10 & -3 \\ -3 & 1 \end{bmatrix}$$

$$\therefore P\left[P'(DD')^{-1}P\right]^{-1}P' =$$

$$\frac{1}{42}\begin{bmatrix} 1 & \cdot \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{bmatrix}\begin{bmatrix} 10 & -3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \cdot & 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$$

$$= \frac{1}{42}\begin{bmatrix} 10 & 7 & 4 & 1 & -2 & -5 & -8 \\ 7 & 5 & 3 & 1 & -1 & -3 & -5 \\ 4 & 3 & 2 & 1 & \cdot & -1 & -2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & \cdot & 1 & 2 & 3 & 4 \\ -5 & -3 & -1 & 1 & 3 & 5 & 7 \\ -8 & -5 & -2 & 1 & 4 & 7 & 10 \end{bmatrix}$$

$$\therefore P\left[P'(DD')^{-1}P\right]^{-1}P'(DD')^{-1}$$

$$= \frac{1}{336}\begin{bmatrix} 10 & 7 & 4 & 1 & -2 & -5 & -8 \\ 7 & 5 & 3 & 1 & -1 & -3 & -5 \\ 4 & 3 & 2 & 1 & \cdot & -1 & -2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & \cdot & 1 & 2 & 3 & 4 \\ -5 & -3 & -1 & 1 & 3 & 5 & 7 \\ -8 & -5 & -2 & 1 & 4 & 7 & 10 \end{bmatrix}\begin{bmatrix} 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 6 & 12 & 10 & 8 & 6 & 4 & 2 \\ 5 & 10 & 15 & 12 & 9 & 6 & 3 \\ 4 & 8 & 12 & 16 & 12 & 8 & 4 \\ 3 & 6 & 9 & 12 & 15 & 10 & 5 \\ 2 & 4 & 6 & 8 & 10 & 12 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{bmatrix}$$

$$= \frac{1}{84}\begin{bmatrix} 28 & 36 & 30 & 16 & \cdot & -12 & -14 \\ 21 & 28 & 25 & 16 & 5 & -4 & -7 \\ 14 & 20 & 20 & 16 & 10 & 4 & \cdot \\ 7 & 12 & 15 & 16 & 15 & 12 & 7 \\ \cdot & 4 & 10 & 16 & 20 & 20 & 14 \\ -7 & -4 & 5 & 16 & 25 & 28 & 21 \\ -14 & -12 & \cdot & 16 & 30 & 36 & 28 \end{bmatrix}$$

Performing this transformation on the first differences 1, 3, 3, 2, 5, 5, 10, we get the smoothed first differences

$$\tfrac{1}{84}(58,\ 147,\ 236,\ 325,\ 414,\ 503,\ 592)$$

which/

which is precisely the same result as that derived in Example I.

Moreover, the transformation

$$DC(D')^{-1} \quad \text{i.e.} \quad DP(P'P)^{-1}P'(D')^{-1}$$

in this example is

$$-\frac{1}{168}\begin{bmatrix} -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 \end{bmatrix}\begin{bmatrix} 119 & 63 & 21 & -7 & -21 & -21 & -7 & 21 \\ 63 & 47 & 33 & 21 & 11 & 3 & -3 & -7 \\ 21 & 33 & 39 & 39 & 33 & 21 & 3 & -21 \\ -7 & 21 & 39 & 47 & 45 & 33 & 11 & -21 \\ -21 & 11 & 33 & 45 & 47 & 39 & 21 & -7 \\ -21 & 3 & 21 & 33 & 39 & 39 & 33 & 21 \\ -7 & -3 & 3 & 11 & 21 & 33 & 47 & 63 \\ 21 & -7 & -21 & -21 & -7 & 21 & 63 & 119 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \cdot & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

$$= \frac{1}{168}\begin{bmatrix} 56 & 16 & -12 & -28 & -32 & -24 & -4 & 28 \\ 42 & 14 & -6 & -18 & -22 & -18 & -6 & 14 \\ 28 & 12 & \cdot & -8 & -12 & -12 & -8 & \cdot \\ 14 & 10 & 6 & 2 & -2 & -6 & -10 & -14 \\ \cdot & 8 & 12 & 12 & 8 & \cdot & -12 & -28 \\ -14 & 6 & 18 & 22 & 18 & 6 & -14 & -42 \\ -28 & 4 & 24 & 32 & 28 & 12 & -16 & -56 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \cdot & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

$$= \frac{1}{84}\begin{bmatrix} 28 & 36 & 30 & 16 & \cdot & -12 & -14 \\ 21 & 28 & 25 & 16 & 5 & -4 & -7 \\ 14 & 20 & 20 & 16 & 10 & 4 & \cdot \\ 7 & 12 & 15 & 16 & 15 & 12 & 7 \\ \cdot & 4 & 10 & 16 & 20 & 20 & 14 \\ -7 & -4 & 5 & 16 & 25 & 28 & 21 \\ -14 & -12 & \cdot & 16 & 30 & 36 & 28 \end{bmatrix}$$

which is the same matrix again as that obtained by using $P[P'(DD')^{-1}P]^{-1}P'(DD')^{-1}$. By yet another method F.M. Harding has also evaluated matrices corresponding to those above. It is a remarkable fact that minimising squared /

squared error of differences should lead to the same result as that by minimising the smoothed data themselves, or the equivalent method of fitting a curve of correspondingly higher degree by the straightforward method of least squares.

CHAPTER II.

LINEAR COMBINATION OF DATA WITH LEAST ERROR USING HARMONIC REPRESENTATION.

9. Let us now consider the Linear Transformation

$$C = P(P'P)^{-1}P'$$

where the representation y = Pa is an Harmonic Function of the form

$$y_x = \frac{1}{\sqrt{2}} a_0 + \sum_{k=1}^{n-1} \left(a_k \cos \frac{k\pi x}{n} + b_k \sin \frac{k\pi x}{n} \right) + \frac{1}{\sqrt{2}} a_n \cos \pi x.$$

If we have 2n data and the representation has k harmonics ($2k < 2n$) then,

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \cos 0 & \sin 0 & \cdots & \cos 0 & \sin 0 \\ \frac{1}{\sqrt{2}} & \cos\frac{\pi}{n} & \sin\frac{\pi}{n} & \cdots & \cos\frac{k\pi}{n} & \sin\frac{k\pi}{n} \\ \frac{1}{\sqrt{2}} & \cos\frac{2\pi}{n} & \sin\frac{2\pi}{n} & \cdots & \cos\frac{2k\pi}{n} & \sin\frac{2k\pi}{n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{\sqrt{2}} & \cos\frac{(2n-1)\pi}{n} & \sin\frac{(2n-1)\pi}{n} & \cdots & \cos\frac{k(2n-1)\pi}{n} & \sin\frac{k(2n-1)\pi}{n} \end{bmatrix}$$

of order $2n \times (2k+1)$

whence $P'P$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{2}} \\ \cos 0 & \cos\frac{\pi}{n} & \cos\frac{2\pi}{n} & \cdots & \cos\frac{(2n-1)\pi}{n} \\ \sin 0 & \sin\frac{\pi}{n} & \sin\frac{2\pi}{n} & \cdots & \sin\frac{(2n-1)\pi}{n} \\ \vdots & \vdots & \vdots & & \vdots \\ \cos 0 & \cos\frac{k\pi}{n} & \cos\frac{2k\pi}{n} & \cdots & \cos\frac{(2n-1)k\pi}{n} \\ \sin 0 & \sin\frac{k\pi}{n} & \sin\frac{2k\pi}{n} & \cdots & \sin\frac{(2n-1)k\pi}{n} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \cos 0 & \sin 0 & \cdots & \cos 0 & \sin 0 \\ \frac{1}{\sqrt{2}} & \cos\frac{\pi}{n} & \sin\frac{\pi}{n} & \cdots & \cos\frac{k\pi}{n} & \sin\frac{k\pi}{n} \\ \frac{1}{\sqrt{2}} & \cos\frac{2\pi}{n} & \sin\frac{2\pi}{n} & \cdots & \cos\frac{2k\pi}{n} & \sin\frac{2k\pi}{n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{\sqrt{2}} & \cos\frac{(2n-1)\pi}{n} & \sin\frac{(2n-1)\pi}{n} & \cdots & \cos\frac{(2n-1)k\pi}{n} & \sin\frac{(2n-1)k\pi}{n} \end{bmatrix}$$

Order $2n \times 2n$.

Denoting /

Denoting the elements of this product by a_{rs} we have

$$a_{11} = n$$

$$a_{1r} = \frac{1}{\sqrt{2}} \sum_{s=0}^{n-1} \cos \frac{rs\pi}{n} \quad \text{or} \quad \frac{1}{\sqrt{2}} \sum_{s=0}^{2n-1} \sin \frac{rs\pi}{n}$$ each of

which is zero. Any other element is of the form

$$\sum_{i=0}^{2n-1} \cos \frac{ir\pi}{n} \cos \frac{is\pi}{n} \quad \text{or} \quad \sum_{i=0}^{2n-1} \sin \frac{ir\pi}{n} \cos \frac{is\pi}{n} \quad \text{or} \quad \sum_{i=0}^{2n-1} \sin \frac{ir\pi}{n} \sin \frac{is\pi}{n}.$$

$$\sum_{i=0}^{2n-1} \cos \frac{ir\pi}{n} \cos \frac{is\pi}{n} = \frac{1}{2} \sum_{i=0}^{2n-1} \cos (r+s) \frac{i\pi}{n} + \cos (r-s) \frac{i\pi}{n}$$

$$= 0 \quad \text{if } r \neq s$$

$$= n \quad \text{if } r = s.$$

$$\sum_{i=0}^{2n-1} \sin \frac{ir\pi}{n} \cos \frac{is\pi}{n} = \frac{1}{2} \sum_{i=0}^{2n-1} \sin (r+s) \frac{i\pi}{n} + \sin (r-s) \frac{i\pi}{n}$$

$= 0$ (In this case r cannot equal s).

$$\sum_{i=0}^{2n-1} \sin \frac{ir\pi}{n} \sin \frac{is\pi}{n} = \frac{1}{2} \sum_{i=0}^{2n-1} \cos (r-s) \frac{i\pi}{n} - \cos (r+s) \frac{i\pi}{n}$$

$$= 0 \quad \text{if } r \neq s$$

$$= n \quad \text{if } r = s.$$

$$\therefore P'P = n \begin{bmatrix} 1 & \cdot & \cdot & \cdot & & \cdot \\ \cdot & 1 & \cdot & \cdot & & \cdot \\ \cdot & \cdot & 1 & \cdot & & \cdot \\ \cdot & \cdot & \cdot & 1 & & \cdot \\ & & & & & \\ \cdot & \cdot & \cdot & & 1 & \cdot \\ \cdot & \cdot & \cdot & & \cdot & 1 \end{bmatrix} \qquad \therefore (P'P)^{-1} = \frac{1}{n} I$$

If we consider the case where k = n the matrix $(P'P)^{-1}$ is still $\frac{1}{n} I$ on account of the factor $\frac{1}{\sqrt{2}}$ in the last term of the representation.

$\therefore\ P(P'P)^{-1}P'$ is

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \cos 0 & \sin 0 & \cdots & \cos 0 & \sin 0 \\ \frac{1}{\sqrt{2}} & \cos\frac{\pi}{n} & \sin\frac{\pi}{n} & \cdots & \cos\frac{k\pi}{n} & \sin\frac{k\pi}{n} \\ \frac{1}{\sqrt{2}} & \cos\frac{2\pi}{n} & \sin\frac{2\pi}{n} & \cdots & \cos\frac{2k\pi}{n} & \sin\frac{2k\pi}{n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{\sqrt{2}} & \cos\frac{(2n-1)\pi}{n} & \sin\frac{(2n-1)\pi}{n} & \cdots & \cos\frac{(2n-1)k\pi}{n} & \sin\frac{(2n-1)k\pi}{n} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{2}} \\ \cos 0 & \cos\frac{\pi}{n} & \cos\frac{2\pi}{n} & \cdots & \cos\frac{(2n-1)\pi}{n} \\ \sin 0 & \sin\frac{\pi}{n} & \sin\frac{2\pi}{n} & \cdots & \sin\frac{(2n-1)\pi}{n} \\ \vdots & \vdots & \vdots & & \vdots \\ \cos 0 & \cos\frac{k\pi}{n} & \cos\frac{2k\pi}{n} & \cdots & \cos\frac{(2n-1)k\pi}{n} \\ \sin 0 & \sin\frac{k\pi}{n} & \sin\frac{2k\pi}{n} & \cdots & \sin\frac{(2n-1)k\pi}{n} \end{bmatrix}$$

Then any element C_{rs} of this matrix product is

$$\frac{1}{n}\left\{\frac{1}{2} + \sum_{j=1}^{k}\left(\cos\frac{j\pi r}{n}\cos\frac{j\pi s}{n} + \sin\frac{j\pi r}{n}\sin\frac{j\pi s}{n}\right)\right\}$$

$$= \frac{1}{n}\left\{\frac{1}{2} + \sum_{j=1}^{k}\cos(r-s)\frac{j\pi}{n}\right\}.$$

If k = n i.e. if we have 2n constants in the representation

$$C_{rs} = \frac{1}{n}\left\{\frac{1}{2} + \sum_{j=1}^{n-1}\cos(r-s)\frac{j\pi}{n} + \frac{1}{2}\cos r\pi\cos s\pi\right\}$$

$$= \frac{1}{n}\left\{\frac{1}{2} + \sum_{j=1}^{n}\cos(r-s)\frac{j\pi}{n} - \frac{1}{2}\cos(r-s)\pi\right\}$$

If $r = s$, $\quad C_{rs} = \frac{1}{n}\left\{\frac{1}{2} + n - \frac{1}{2}\right\} = 1$

If $r \neq s$ and $(r-s)$ is even,

$$C_{rs} = \frac{1}{n}\left\{\frac{1}{2} + 0 - \frac{1}{2}\right\} = 0$$

If $r \neq s$ and $(r-s)$ is odd,

$$C_{rs} = \frac{1}{n}\left\{\frac{1}{2} - 1 + \frac{1}{2}\right\} = 0$$

whence $C = I$

Thus the matrix $P(P'P)^{-1}P'$ is completely determined for all values of $2k \leqslant 2n$.

10. PROPERTIES OF C.

I. **C. is axi-symmetric**

$$C_{rs} = \frac{1}{n}\left\{\frac{1}{2} + \sum_{j=1}^{k} \cos(r-s)\frac{j\pi}{n}\right\} = C_{sr}$$

This is also obvious from the fact that

$$\frac{1}{n}[PP']' = \frac{1}{n}(P')'P' = \frac{1}{n}PP'.$$

II. **C. is centro-symmetric.**

$$\begin{aligned} C_{n-r,\,s-r} &= \frac{1}{n}\left\{\frac{1}{2} + \sum_{j=1}^{k} \cos(\overline{n-r} - \overline{n-s})\frac{j\pi}{n}\right\} \\ &= \frac{1}{n}\left\{\frac{1}{2} + \sum_{j=1}^{k} \cos(s-r)\frac{j\pi}{n}\right\} \\ &= C_{sr} = C_{rs}. \end{aligned}$$

III. **C. is per-symmetric and alternant.**

$$\begin{aligned} C_{r-1,\,s-1} &= \frac{1}{n}\left\{\frac{1}{2} + \sum_{j=1}^{k} \cos(\overline{r-1} - \overline{s-1})\frac{j\pi}{n}\right\} \\ &= \frac{1}{n}\left\{\frac{1}{2} + \sum_{j=1}^{k} \cos(r-s)\frac{j\pi}{n}\right\} \\ &= C_{rs} \end{aligned}$$

and

$$\begin{aligned} C_{r,\,2n} &= \frac{1}{n}\left\{\frac{1}{2} + \sum_{j=1}^{k} \cos(r-2n)\frac{j\pi}{n}\right\} \\ &= \frac{1}{n}\left\{\frac{1}{2} + \sum_{j=1}^{k} \cos\left(\frac{rj\pi}{n} - j.2\pi\right)\right\} \\ &= \frac{1}{n}\left\{\frac{1}{2} + \sum_{j=1}^{k} \cos\frac{rj\pi}{n}\right\} \\ &= \frac{1}{n}\left\{\frac{1}{2} + \sum_{j=1}^{k} \cos(\overline{r+1} - 1)\frac{j\pi}{n}\right\} \\ &= C_{r+1,\,1}. \end{aligned}$$

IV. **C. is idempotent**

Any element $(C^2)_{rs}$ of C^2 is obtained by multiplying the s^{th} column of C by the r^{th} row of C.

IV. contd.

$$\therefore (C^2)_{rs} = \sum_{v=1}^{2n} \frac{1}{n^2}\left[\frac{1}{2} + \sum_{j=1}^{k} \cos(r-v)\frac{j\pi}{n}\right]\left[\frac{1}{2} + \sum_{j=1}^{k} \cos(v-s)\frac{j\pi}{n}\right]$$

$$= \sum_{v=1}^{2n} \frac{1}{n^2}\left[\frac{1}{4} + \frac{1}{2}\left(\sum_{j=1}^{k} \cos(r-v)\frac{j\pi}{n} + \cos(v-s)\frac{j\pi}{n}\right) + \sum_{j=1}^{k}\cos(r-v)\frac{j\pi}{n}\sum_{j=1}^{k}\cos(v-s)\frac{j\pi}{n}\right]$$

$$= \sum_{v=1}^{2n} \frac{1}{n^2}\left[\frac{1}{4} + \sum_{j=1}^{k} \cos\left(\frac{r-s}{2}\right)\frac{j\pi}{n}\cos\left(\frac{r-2v+s}{2}\right)\frac{j\pi}{n} + \sum_{i=1}^{k}\sum_{j=1}^{k}\cos(r-v)\frac{i\pi}{n}\cos(v-s)\frac{j\pi}{n}\right]$$

$$= \frac{2n}{4n^2} + 0 + \frac{1}{n^2}\sum_{v=1}^{2n}\sum_{i=1}^{k}\sum_{j=1}^{k}\cos(r-v)\frac{i\pi}{n}\cos(v-s)\frac{j\pi}{n}$$

$$= \frac{1}{2n} + \frac{1}{2}\cdot\frac{1}{n^2}\sum_{v=1}^{2n}\sum_{i=1}^{k}\sum_{j=1}^{k}\left\{\cos[ir-js-v(i-j)]\frac{\pi}{n} + \cos[ir+js-v(i+j)]\frac{\pi}{n}\right\}$$

$$= \frac{1}{2n} + \frac{1}{2n^2}\sum_{i=1}^{k}\sum_{j=1}^{k}\left\{\sum_{v=1}^{2n}\cos[ir-js-v(i-j)]\frac{\pi}{n} + \cos[ir+js-v(i+j)]\frac{\pi}{n}\right\}$$

When $i \neq j$ the second term vanishes.

$$\therefore (C^2)_{rs} = \frac{1}{2n} + \frac{1}{2n^2}\sum_{j=1}^{k}\sum_{v=1}^{2n}\cos(r-s)\frac{j\pi}{n} + 0$$

$$= \frac{1}{2n} + \frac{1}{2n^2}\cdot 2n\sum_{j=1}^{k}\cos(r-s)\frac{j\pi}{n}$$

$$= \frac{1}{n}\left\{\frac{1}{2} + \sum_{j=1}^{k}\cos(r-s)\frac{j\pi}{n}\right.$$

$$= C_{rs}$$

$\therefore C^2 = C \quad \therefore C^n = C$ i.e. <u>C is idempotent.</u>

This theorem may also be proved thus

$$C = \frac{1}{n}PP'$$

$$\therefore C^2 = \frac{1}{n}PP'.\frac{1}{n}PP'$$

$$= \frac{1}{n^2}PP'PP'$$

$$= \frac{1}{n^2}P.\,nI.\,P'$$

$$= \frac{1}{n}PP'$$

$$= C.$$

V. /

V. The sum of the elements of any row or column of C is unity.

Sum of elements in r^{th} row

$$\sum_{s=1}^{2n} \frac{1}{n}\left\{\frac{1}{2}+\sum_{j=1}^{k} \cos(r-s)\frac{j\pi}{n}\right\}$$

$$= 1 + \sum_{s=1}^{2n}\sum_{j=1}^{k} \cos(r-s)\frac{j\pi}{n}$$

$$= 1 + \sum_{j=1}^{k}\sum_{s=1}^{2n} \cos(r-s)\frac{j\pi}{n}$$

$$= 1 + 0$$

$$= 1$$

VI. The sum of the elements in the leading diagonal is $(2k + 1)$

Since C is persymmetric each element in the leading diagonal is equal to C_{11}

and $$C_{11} = \frac{1}{n}\left\{\frac{1}{2}+\sum_{j=1}^{k}\cos(1-1)\frac{j\pi}{n}\right\}$$

$$= \frac{1}{n}\left\{\frac{1}{2}+k\right\}$$

$\therefore$ Sum of Diagonal Elements

$$= 2n \cdot \frac{1}{n}\left\{\frac{1}{2}+k\right\}$$

$$= 2k+1.$$

VII. The sum of the squares of the elements in any row is $\frac{1}{2n}(2k + 1)$.

From theorem IV the sum of the squares of the elements in the r^{th} row is

$$C_{rr}$$

$$= C_{11}$$

$$= \frac{1}{2n}(2k+1)$$

VIII. The sum of the elements in the secondary diagonal is unity.

Since C is persymmetric the elements in the secondary diagonal are $C_{1,2n}, \quad C_{1,2n-2}$, etc.,

$$\therefore \text{Sum} = 2\left(C_{1,2n} + C_{1,2n-2} + \cdots + C_{1,6} + C_{1,4} + C_{12}\right)$$

$$= 2\sum_{s=1}^{n} \frac{1}{n}\left[\frac{1}{2} + \sum_{j=1}^{k}\cos(1-2s)\frac{j\pi}{n}\right]$$

$$= 1 + \frac{2}{n}\sum_{s=1}^{n}\sum_{j=1}^{k}\cos(1-2s)j\frac{\pi}{n}$$

$$= 1 + \frac{2}{n}\sum_{j=1}^{k}\sum_{s=1}^{n}\cos(1-2s)j\frac{\pi}{n}$$

$$= 1 + 0$$

IX. The Latent Roots of C are units and zeros.

Since every square matrix satisfies its own characteristic equation and $C^2 = C$ (Th. IV), C satisfies the reduced characteristic equation

$$\lambda^2 - \lambda = 0$$

$\therefore \lambda = 0$ or $\lambda = 1$ in certain multiplicities.

Since C is of order 2n x 2n, the characteristic equation is of degree 2n.

Sum of the roots is the leading diagonal elements

$(2k + 1)$ (Th. VI).

$\therefore$ Latent roots are $(2k + 1)$ units

and $(2n - 2k - 1)$ zeros.

X. The rank of C is 2k + 1.

This follows at once since the representation contains $(2k + 1)$ constants $a_0, a_1, b_1, \ldots\ldots\ldots a_k, b_k$ which are determined by using any $(2k + 1)$ of the arbitrary u's. Any other row, therefore, of C after the $(2k + 1)^{th}$ is a linear combination of the first $(2k + 1)$ rows.

XI. The sum of the principal diagonal minors of C of order $S > 2k + 1$ is zero.

The characteristic equation of C may be written

$$f(\lambda) = \lambda^{2n} + a_1\lambda^{2n-1} + \cdots + a_s\lambda^{2n-s} + \cdots + a_{2n} = 0$$

where a_s = sum of diagonal minors of order s of C.

but $(-1)^s a_s$ = sum of roots of $f(\lambda) = 0$ taken s at a time.

But /

But each group of s roots contains at least $s-\overline{2k+1}$ zeros.
whence a_s must always be zero for $s > 2k+1$.
In the same way $a_{s+1}, a_{s+2}, \dots\dots\dots a_{2n}$ are all zero.

XII. The matrix C. is singular.

$|C|$ is the product of the 2n latent roots of C

But some of the latent roots are zeros

$$\therefore |C| = 0$$

XIII. In any row of C the sum of the elements in the odd places is equal to the sum of the elements in the even places, each being equal to ½.

The sum of the elements in the odd places of the first row of C is

$$\frac{1}{n}\sum_{s=1}^{n}\left\{\frac{1}{2} + \sum_{j=1}^{k}\cos(1-\overline{2s-1})\frac{j\pi}{n}\right\}$$

$$= \frac{1}{2} + \sum_{s=1}^{n}\sum_{j=1}^{k}\cos(1-s)\frac{2j\pi}{n}$$

$$= \frac{1}{2} \quad \text{since} \quad \sum_{s=1}^{n}\cos(1-s)\frac{2j\pi}{n} = 0$$

But the sum of all the elements in the first row is unity.

$\therefore$ Sum of elements occupying even places is also ½.

XIV. The adjugate matrix of C is a multiple of

$$\begin{bmatrix} 1 & -1 & 1 & -1 & \cdots\cdots & -1 \\ -1 & 1 & -1 & 1 & \cdots\cdots & 1 \\ 1 & -1 & 1 & -1 & \cdots\cdots & -1 \\ \vdots & & & & & \vdots \\ -1 & 1 & -1 & 1 & \cdots\cdots & 1 \end{bmatrix}$$ of order 2n x 2n.

Since /

Since the product of a singular matrix and its adjugate is the zero matrix, the theorem follows at once since the sum of the elements in the odd places of any row of C is equal to the sum of the elements in the even places.

PRACTICAL FOURIER ANALYSIS

11. In practical Fourier Analysis a common number of data to consider is 12 or 24.

We shall evaluate $P(P'P)^{-'}P'$ for case where $2n = 12$

In this case the representation is

$$\frac{1}{\sqrt{2}} a_0 + a_1 \cos\frac{2\pi x}{12} + \cdots + a_k \cos\frac{k.2\pi x}{12} + \cdots + \frac{1}{\sqrt{2}} a_6 \cos \pi x.$$
$$+ b_1 \sin\frac{2\pi x}{12} + \cdots + b_k \sin\frac{k.2\pi x}{12} + \cdots$$

and we shall consider $k = 0,1,2,3,\ldots 5$, and finally the whole representation including the term in a_6.

$(P'P)^{-'} = \frac{1}{6} I$ for all values of $k = 0,1,2.\ldots.\ 6.$

$$\Phi = \begin{bmatrix}
\frac{1}{\sqrt{2}} & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} & \cdot & 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2} & -1 & \cdot & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \cdot & 1 & -1 & \cdot & \cdot & -1 & 1 & \cdot & \cdot & 1 & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 1 & \cdot & -\frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{3}}{2} & \cdot & 1 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -1 & \cdot & 1 & \cdot & -1 & \cdot & 1 & \cdot & -1 & \cdot & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} & \cdot & -1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 1 & \cdot & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \cdot & -1 & -1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & -1 & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & -1 & \cdot & -\frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{3}}{2} & \cdot & -1 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}}
\end{bmatrix}$$

Since /

Since C is persymmetric we need only calculate its top row.

If $k = 0$, P simply consists of the first column of Φ i.e. the column vector $\{\frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}} \cdots\cdots \frac{1}{\sqrt{2}}\}$ hence the top row of $P P'$ is $[\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2} \cdots\cdots \frac{1}{2}]$ and top row of C is $\frac{1}{6}[\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}]$

If $k = 1$, P consists of the first three columns of Φ so that $P P'$ is obtained at once from the previous top row by simply adding to each the elements of the second row of P', that is, the second column of P

$\therefore$ Top row of C is

$$\frac{1}{6}\left[\tfrac{1}{2}+1\quad \tfrac{1}{2}+\tfrac{\sqrt{3}}{2}\quad 1\quad \tfrac{1}{2}\quad \cdot\quad \tfrac{1}{2}-\tfrac{\sqrt{3}}{2}\quad -\tfrac{1}{2}\quad \tfrac{1}{2}-\tfrac{\sqrt{3}}{2}\quad \cdot\quad \tfrac{1}{2}\quad 1\quad \tfrac{1}{2}+\tfrac{\sqrt{3}}{2}\right]$$

Proceeding in this way we evaluate the top rows for $k = 2,3,4$, and 5. These are

$$\frac{1}{6}\left[\tfrac{1}{2}+2\quad 1+\tfrac{\sqrt{3}}{2}\quad \tfrac{1}{2}\quad -\tfrac{1}{2}\quad -\tfrac{1}{2}\quad 1-\tfrac{\sqrt{3}}{2}\quad \tfrac{1}{2}\quad 1-\tfrac{\sqrt{3}}{2}\quad -\tfrac{1}{2}\quad -\tfrac{1}{2}\quad \tfrac{1}{2}\quad 1+\tfrac{\sqrt{3}}{2}\right]$$

$$\frac{1}{6}\left[\tfrac{1}{2}+3\quad 1+\tfrac{\sqrt{3}}{2}\quad -\tfrac{1}{2}\quad -\tfrac{1}{2}\quad \tfrac{1}{2}\quad 1-\tfrac{\sqrt{3}}{2}\quad -\tfrac{1}{2}\quad 1-\tfrac{\sqrt{3}}{2}\quad \tfrac{1}{2}\quad -\tfrac{1}{2}\quad -\tfrac{1}{2}\quad 1+\tfrac{\sqrt{3}}{2}\right]$$

$$\frac{1}{6}\left[\tfrac{1}{2}+4\quad \tfrac{1}{2}+\tfrac{\sqrt{3}}{2}\quad -1\quad \tfrac{1}{2}\quad \cdot\quad \tfrac{1}{2}-\tfrac{\sqrt{3}}{2}\quad \tfrac{1}{2}\quad \tfrac{1}{2}-\tfrac{\sqrt{3}}{2}\quad \cdot\quad \tfrac{1}{2}\quad -1\quad \tfrac{1}{2}+\tfrac{\sqrt{3}}{2}\right]$$

$$\frac{1}{6}\left[\tfrac{1}{2}+5\quad \tfrac{1}{2}\quad -\tfrac{1}{2}\quad \tfrac{1}{2}\quad -\tfrac{1}{2}\quad \tfrac{1}{2}\quad -\tfrac{1}{2}\quad \tfrac{1}{2}\quad -\tfrac{1}{2}\quad \tfrac{1}{2}\quad -\tfrac{1}{2}\quad \tfrac{1}{2}\right]$$

Finally to establish the top row for $k = 6$, that is when we take all the elements of Φ, we add to each element of the last case $\frac{1}{\sqrt{2}}$ times the last row of P' i.e. the last column of P. It is

$$\frac{1}{6}\left[6\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\right]$$

In this manner, matrices have been formed for those cases where the number of data 2n is 4,6,8,10,12, 16 and 24, and for values of $k = 0,1,2 \ldots\ldots n$. These appear in the appendix in two forms; the surd form for theoretical /

theoretical consideration and the decimal form for rapid calculation by means of a machine.

12. <u>Relationship between graduated values for different values of k.</u> These relationships are of little practical value, but could be used as a check.

Let ${}_s y_r$ denote the graduated value of u_r for $k = s$

Operating on the observed data, u, by C we get,

$$k = 1, \quad {}_1y_0 = \tfrac{1}{12}\left\{3u_0 + \Sigma' u_{odd} + \sqrt{3}(u_1+u_{11}-u_5-u_7) + 2(u_2+u_{10}) - u_6\right\}$$

$$k = 2, \quad {}_2y_0 = \tfrac{1}{12}\left\{5u_0 + (u_2+u_6+u_{10}) - (u_3+u_4+u_8+u_9) + 2(u_1+u_5+u_7+u_{11}) + \sqrt{3}(u_1+u_{11}-u_5-u_7)\right\}$$

$$k = 3, \quad {}_3y_0 = \tfrac{1}{12}\left\{7u_0 - (u_2+u_3+u_6+u_9+u_{10}) + (u_4+u_8) + 2(u_1+u_5+u_7+u_{11}) + \sqrt{3}(u_1+u_{11}-u_5-u_7)\right\}$$

$$k = 4, \quad {}_4y_0 = \tfrac{1}{12}\left\{9u_0 + \Sigma' u_{odd} + \sqrt{3}(u_1+u_{11}-u_5-u_7) - 2(u_2+u_{10}) + u_6\right\}$$

$$k = 5, \quad {}_5y_0 = \tfrac{1}{12}\left\{12u_0 - \Sigma' u_{even} + \Sigma' u_{odd}\right\}$$

$$k = 6, \quad {}_6y_0 = u_0$$

Having established the graduated values for the case k = 1, let us examine what terms must be added to give the graduated values for k = 2,3 and 4. From the above we obtain $${}_2y_0 = {}_1y_0 + \tfrac{1}{12}\left\{\Sigma' u_{odd} - \Sigma' u_{even} + 3(u_0+u_6-u_3-u_9)\right\}$$

Hence to evaluate ${}_2y_s$ for S = 0 to S = 11, the only additional expressions to be calculated are

$$u_0 + u_6 - u_3 - u_9$$
$$u_1 + u_7 - u_4 - u_{10}$$
$$u_2 + u_8 - u_5 - u_{11}$$

$${}_3y_0 = {}_2y_0 + \tfrac{1}{6}\left\{(u_0+u_4+u_8) - (u_2+u_6+u_{10})\right\}$$

$\therefore$ two additional expressions to be calculated.

$${}_{4}y_0 = {}_{3}y_0 + \frac{1}{12}\left\{3(u_0 + u_3 + u_6 + u_9) - \Sigma' u_n\right\}$$

$\therefore$ three additional expressions to be calculated.

$${}_{5}y_0 = u_0 + \frac{1}{12}\left\{\Sigma' u_{odd} - \Sigma' u_{even}\right\}$$

EXAMPLE.

The following table shows the graduation of the data 7.1 6.7 6.0 5.2 4.6 4.6 5.1 5.7 6.2 6.6 6.8 7.0 for different values of k. The smoothed values were obtained by operating on the observed data with the matrices for 2n = 12, k = 1,2,3,4,5 and 6, and checked by the method of the last paragraph.

UNGRADUATED		GRADUATED					
		k = 1	k = 2	k = 3	k = 4	k = 5	k = 6
u_0	7.1	6.957	7.057		7.091		
u_1	6.7	6.476	6.726		6.709		
u_2	6.0	5.859	6.009		5.992		
u_3	5.2	5.270	5.170	Same	5.204	Same	Same
u_4	4.6	4.868	4.618	as	4.611	as	as
u_5	4.6	4.761	4.611	k = 2	4.594	Original	Original
u_6	5.1	4.976	5.076		5.110	Ungrad.	Ungrad.
u_7	5.7	5.457	5.707		5.690	Data	Data
u_8	6.2	6.074	6.224		6.207		
u_9	6.6	6.663	6.563		6.597		
u_{10}	6.8	7.065	6.815		6.798		
u_{11}	7.0	7.173	7.023		7.006		

By /

13. By means of the Fourier Analysis Form which Whittaker and Robinson[†] suggest using the harmonic function which is determined by these 12 data is

$$y = 5\cdot967 + 0\cdot991\cos\theta + 0\cdot1\cos 2\theta + 0\cdot033\cos 4\theta + 0\cdot009\cos 5\theta$$
$$+ 0\cdot696\sin\theta + 0\cdot231\sin 2\theta \qquad - 0\cdot004\sin 5\theta$$

The coefficients $a_0, a_1, b_1, \ldots$ of the representation may be determined directly in the following manner.

Since the normal equations for these coefficients are given by $a = (P'P)^{-1}P'u$ [*]

$$\text{and} \quad (P'P)^{-1} = \tfrac{1}{6}I$$

$$\therefore a = \tfrac{1}{6}P'u.$$

$$\therefore a = \frac{1}{12}\begin{bmatrix}
\sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 \\
2 & \sqrt3 & 1 & \cdot & -1 & -\sqrt3 & -2 & -\sqrt3 & -1 & \cdot & 1 & \sqrt3 \\
\cdot & 1 & \sqrt3 & 2 & \sqrt3 & 1 & \cdot & -1 & -\sqrt3 & -2 & -\sqrt3 & -1 \\
2 & 1 & -1 & -2 & -1 & 1 & 2 & 1 & -1 & -2 & -1 & 1 \\
\cdot & \sqrt3 & \sqrt3 & \cdot & -\sqrt3 & -\sqrt3 & \cdot & \sqrt3 & \sqrt3 & \cdot & -\sqrt3 & -\sqrt3 \\
2 & \cdot & -2 & \cdot & 2 & \cdot & -2 & \cdot & 2 & \cdot & -2 & \cdot \\
\cdot & 2 & \cdot & -2 & \cdot & 2 & \cdot & -2 & \cdot & 2 & \cdot & -2 \\
2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 & -1 \\
\cdot & \sqrt3 & -\sqrt3 & \cdot & \sqrt3 & -\sqrt3 & \cdot & \sqrt3 & -\sqrt3 & \cdot & \sqrt3 & -\sqrt3 \\
2 & -\sqrt3 & 1 & \cdot & -1 & \sqrt3 & -2 & \sqrt3 & -1 & \cdot & 1 & -\sqrt3 \\
\cdot & 1 & -\sqrt3 & 2 & -\sqrt3 & 1 & \cdot & -1 & \sqrt3 & -2 & \sqrt3 & -1 \\
\sqrt2 & -\sqrt2 & \sqrt2 & -\sqrt2 & \sqrt2 & -\sqrt2 & \sqrt2 & -\sqrt2 & \sqrt2 & -\sqrt2 & \sqrt2 & -\sqrt2
\end{bmatrix}
\begin{bmatrix} 7\cdot1 \\ 6\cdot7 \\ 6\cdot0 \\ 5\cdot2 \\ 4\cdot6 \\ 4\cdot6 \\ 5\cdot1 \\ 5\cdot7 \\ 6\cdot2 \\ 6\cdot6 \\ 6\cdot8 \\ 7\cdot0 \end{bmatrix}
=
\begin{bmatrix} 5\cdot966\ 667 \\ 0\cdot990\ 748 \\ 0\cdot696\ 410 \\ 0\cdot1 \\ 0\cdot230\ 940 \\ 0 \\ 0 \\ 0\cdot033\ 333 \\ 0 \\ 0\cdot009\ 252 \\ 0\cdot003\ 590 \\ 0 \end{bmatrix}$$

The second method is obviously useful when using a calculating machine. Moreover, if we are fitting our data to a curve with a specified number of harmonics we need only premultiply u by the appropriate number of rows of P'. The matrices from which the coefficients are /

† The Calculus of Observations, chap. 10.

* A.C. Aitken, Proc. Roy. Soc. Edin., Vol. 55, p. 43.

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Harmonic Analysis of .... Period ....

θ.				
0°	u_0	7.1		
30°	u_1	6.7		
60°	u_2	6.0		
90°	u_3	5.2		
120°	u_4	4.6		
150°	u_5	4.6		
180°	u_6	5.1		
210°	u_7	5.7		
240°	u_8	6.2		
270°	u_9	6.6		
300°	u_{10}	6.8		
330°	u_{11}	7.0		

(u_0 to u_6)	7.1	6.7	6.0	5.2	4.6	4.6	5.1
(u_{11} to u_7)		7.0	6.8	6.6	6.2	5.7	
Sums (v_0 to v_6)	7.1	13.7	12.8	11.8	10.8	10.3	5.1
Diffs. (w_1 to w_5)		-0.3	-0.8	-1.4	-1.6	-1.1	

(v_0 to v_3)	7.1	13.7	12.8	11.8
(v_6 to v_4)	5.1	10.3	10.8	
Sums (p_0 to p_3)	12.2	24.0	23.6	11.8
Diffs. (q_0 to q_2)	2.0	3.4	2.0	

(w_1 to w_3)	-0.3	-0.8	-1.4
(w_5 to w_4)	-1.1	-1.6	
Sums (r_1 to r_3)	-1.4	-2.4	-1.4
Diffs. (s_1 to s_2)	+0.8	+0.8	

	$p_1 =$ 24.0	$p_2 =$ 23.6	$q_2 =$ 2.0	$r_1 =$ -1.4
($\frac{1}{2}$ line above)	$h_1 =$ 12.0	$h_2 =$ 11.8	$l_2 =$ 1.0	$m_1 =$ -0.7

	$q_1 =$ 3.4	$r_2 =$ -2.4	$s_1 =$ 0.8	$s_2 =$ 0.8
(0·866 × line above)	$l_1 =$ 2.944	$m_2 =$ -2.078	$n_1 =$ 0.693	$n_2 =$ 0.693

	$p_0 =$ 12.2 $p_1 =$ 24.0 $p_2 =$ 23.6 $p_3 =$ 11.8	$q_0 =$ 2.0 $l_1 =$ 2.944 $l_2 =$ 1.0	$p_0 =$ 12.2 $h_1 =$ 12.0 $p_3 =$ 11.8 $h_2 =$ 11.8	$p_0 =$ 12.2 $p_3 =$ 11.8 $h_1 =$ 12.0 $h_2 =$ 11.8	$m_1 =$ -0.7 $m_2 =$ -2.078 $r_3 =$ -1.4	$q_0 =$ 2.0 $q_2 =$ 2.0	$n_1 =$ 0.693 $n_2 =$ 0.693	$r_1 =$ -1.4 $r_3 =$ -1.4
Sum of 1st column .	35.8	3.0	24.0	24.2	-2.1			
Sum of 2nd column	35.8	2.944	23.8	23.6	-2.078			
Sum	71.6 $= 12a_0$	5.944 $= 6a_1$			-4.178 $= 6b_1$		1.386 $= 6b_2$	
Diff.	0 $= 12a_6$	0.056 $= 6a_5$	0.2 $= 6a_4$	0.6 $= 6a_2$	-0.022 $= 6b_5$	0 $= 6a_3$	0 $= 6b_4$	0 $= 6b_3$

RESULT: $u = a_0 + a_1 \cos\theta + a_2 \cos 2\theta + a_3 \cos 3\theta + a_4 \cos 4\theta + a_5 \cos 5\theta + a_6 \cos 6\theta$
$+ b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta + b_4 \sin 4\theta + b_5 \sin 5\theta$
$= 5.967 + 0.991 \cos\theta + 0.1 \cos 2\theta + 0.033 \cos 4\theta + 0.009 \cos 5\theta$
$- 0.696 \sin\theta + 0.231 \sin 2\theta + 0.000 \sin 4\theta - 0.004 \sin 5\theta$

CHECKS: $u_0 = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6$
$w_1 = b_1 + 2b_3 + b_5 + 1{\cdot}732(b_2 + b_4)$

are derived are also given in the appendix.

We may therefore, either determine the graduated values directly, or first of all find the harmonic series for the graduated values y and evaluate y for $\theta = 0, \frac{2\pi}{12}$ etc. For example, if we wish our smoothed function to include harmonics up to the third, then we premultiply u by the matrix with k = 3, and so obtain the graduated values. These graduated values are the same as those obtained from the actual harmonic series which includes the terms as far as a_3 and b_3. Thus the smoothed results for k = 3 in the example above could have been determined from the series

$$y = 5.967 + 0.991\cos\theta + 0.1\cos 2\theta + 0.0\cos 3\theta + 0.696\sin\theta + 0.231\sin 2\theta + 0.0\sin 3\theta$$

The question now arises, how many terms of the harmonic series, or what value of k, will constitute what may fairly be regarded as a good fit, or indeed the most probable fit to the given crude data. This leads us to discuss the question of errors.

ERRORS AND VARIANCES

14. One of our fundamental postulates is that each of the given data is subject to the same standard error σ, and before we can say whether a fit is good or not, σ must be known. After the observations have been graduated we may compare σ with the mean residual standard error and so estimate the goodness of fit. Should the mean residual standard error be very small compared with σ, then the fitting has been too accurate; indeed by introducing /

introducing 2n constants in the representation of 2n data, the mean residual standard error would be zero. In our subsequent discussion we shall assume that σ is unity.

Variance and Product Variances of Graduated Data.

Since the graduated values y are obtained from the observed data u by means of the linear combination y = Cu, the variances and product variances of the 2n smoothed values are the elements in the variance matrix

$V = CC'$ where $C = P(P'P)^{-1}P'$ and P is the matrix associated with the harmonic function above.

Since $C' = C$ and $C^2 = C$

then $V = C$

so that the variance matrix is identical with the matrix producing the smoothed values, and the properties of this matrix have already been investigated. The variances of the fitted values are the elements in the leading diagonal of V, and since V is persymmetric, all the fitted values will have the same variance $\frac{2k+1}{2n}$, where 2n is the number of data and 2k + 1 the number of constants in the harmonic function.

Also, the product variance of two fitted values y_r and y_s will be the element V_{rs} of V or c_{rs} of C namely $\frac{1}{n}\left\{\frac{1}{2}+\sum_{j=1}^{k}\cos(r-s)\frac{j\pi}{n}\right\}$

15. Total Variance of Residuals.

Since the smoothed values y of the observed data u are given by y = Pa, the sum of squared residuals after fitting has been done is given by

$$s^2 = (u-Pa)'(u-Pa)$$
$$= u'u - u'Pa - a'P'u + a'P'Pa$$

but /

but on account of the minimal condition $\frac{\partial s^2}{\partial a} = 0$

it follows that $P'Pa = P'u$

$$\therefore\ s^2 = u'u - a'P'Pa - a'P'Pa + a'P'Pa$$

$$= u'u - a'P'Pa$$

Now $u'u$ is simply $\sum_{j=1}^{2n} u_j^2$

and $a'P'Pa = a'\,nI\,a$

$$= n\,a'a$$

$$= na_0^2 + n\sum_{r=1}^{k} a_r^2 + b_r^2$$

$$\therefore\ s^2 = \sum_{j=1}^{2n} u_j^2 - na_0^2 - n\sum_{r=1}^{k} a_r^2 + b_r^2$$

Mean Variance of Residuals.

16. In order to estimate the mean variance of residuals, which would be comparable with the error variance of one observation, we must divide the total variance of residuals s^2 by 2n - (2k + 1), the number of independent fitted data.

Again, since y = Cu

the residuals $u - y = u - Cu = (I - C)u$

$\therefore$ Residual variance matrix is $(I-C)'(I-C)$

$$= I - C - C' + C'C$$

$= I - C$ since C is symmetric and idempotent.

It was proved above that the trace of C is (2k + 1)

$\therefore$ Trace of $I - C$ is $2n - (2k+1)$

$$= 2n - 2k - 1$$

which gives the measure of the sum of the residual variances; if the observed data u have each variance σ^2, the sum of residual variances is $(2n - 2k - 1)\sigma^2$. Hence to estimate the goodness of fit we compare $\frac{s^2}{2n-2k-1}$ with σ^2.

For /

For the example given above in para. 12 the sum of squares of residuals was computed from

$$S^2 = \sum_{j=1}^{2n} u_j^2 - n a_0^2 - n \sum_{r=1}^{k} a_r^2 + b_r^2$$

$k = 1, \quad S^2 = 0 \cdot 3874$

$k = 2, \quad S^2 = 0 \cdot 0074$

$k = 3, \quad S^2 = 0 \cdot 0074$

$k = 4, \quad S^2 = 0 \cdot 0007$

$k = 5, \quad S^2 = 0$

Dividing these sum squares of residuals by 2n - 2k - 1 the following mean residual variances were obtained

k = 1, 0.0430 k = 2, 0.0011

k = 3, 0.0015 k = 4, 0.0002

If the value of σ^2 in this example is 0.0022, then the means residual errors would be approximately

$$20\sigma^2, \quad \tfrac{1}{2}\sigma^2, \quad \tfrac{2}{3}\sigma^2, \quad \tfrac{1}{10}\sigma^2.$$

We may expect therefore that in this case the most reliable fit is obtained by giving k the value 2 or 3.

Applying Pearson's χ^2 test we find that for

$k = 1, \quad \chi^2 = \dfrac{0 \cdot 3874}{0 \cdot 0022} = 177$ for 9 degrees of freedom.

$k = 2, \quad \chi^2 = \dfrac{0.0074}{0 \cdot 0022} = 3 \cdot 4$ for 7 degrees of freedom.

$k = 3, \quad \chi^2 = \dfrac{0 \cdot 0074}{0 \cdot 0022} = 3 \cdot 4$ for 5 degrees of freedom.

$k = 4, \quad \chi^2 = \dfrac{0.0007}{0 \cdot 0022} = 0 \cdot 32$ for 3 degrees of freedom.

From the χ^2 table given by R.A. Fisher※, we find the values of P, whose position in the probability curve indicates the goodness of fit, are 0.01, 0.84, 0.76 and 0.956 respectively. From these again we see that the best fits are given by k = 2 and k = 3.

Properties /

※ Statistical Methods for Research Workers (1936).

Properties of the Residual Matrix I - C.

17. Since the elements in the leading diagonal of I - C are given by $1 - C_{rr}$, where C_{rr} is the corresponding element in C, and each non-diagonal element of I - C is given by the corresponding $-C_{rs}$, the proofs of the properties of I - C follow at once from the properties of C.

I. I - C is axi-symmetric.

II. I - C is centro-symmetric.

III. I - C is persymmetric and alternant.

IV. I - C is idempotent

$$\text{for } (I - C)^2 = I - 2C + C^2$$

$$= I - 2C + C \text{ since } C \text{ is idempotent.}$$

$$= I - C, \text{ whence the theorem.}$$

V. The sum of any row or column of I - C is zero.

VI. The trace of I - C is 2n - 2k - 1

Each diagonal element is $1 - \frac{1}{2n}(2k + 1)$

VII. The sum of the elements in the secondary diagonal is zero.

VIII. The sum of the squares of the elements of any row is $\frac{1}{2n}(2n - 2k - 1)$.

IX. The latent roots of I - C are units and zeros, there being (2n - 2k - 1) units and (2k + 1) zeros.

X. The sum of the principal diagonal minors of I - C of order $S > 2n - 2k - 1$ is zero.

XI. /

<u>XI.</u> In any row or column of I - C, the sum of the elements in the odd or even places is $\pm \frac{1}{2}$ according as the diagonal element of that row or column occupies an odd or even place.

<u>XII.</u> I - C is a singular matrix.

<u>XIII.</u> The adjugate matrix of I - C is a multiple of the matrix of order 2n all of whose elements are units.

<u>EXAMPLE II, using 24 data.</u>

18. The data for the following example were obtained from the British Polar Year Expedition, Fort Rae (1932-33), Vol. II. Hourly observations were made of the horizontal component of the earth's magnetic field over a period of thirteen months. These are given in γ^2 units and averaged for each hour of the day. These averaged observations vary from 39 units to 141 units with the rather large standard deviation of 14, due to the fact that the observations were subject to large magnetic disturbances owing to the proximity of Fort Rae to the North magnetic pole.

43 41 45 50 61 91 117 141 125 121 96 83

82 81 79 77 63 55 43 41 39 42 45 43.

Before proceeding to graduate these data, we shall examine in the first instance what value of <u>k</u> gives the best fit.

$$\sum u_n^2 = 143,136$$

Premultiplying /

Premultiplying the data by the matrix producing the Fourier coefficients (Appendix, page 85) for the case where $2n = 24$ the following are the values for a_0, a_1, b_1, a_2, b_2, a_3, b_3, only the top 7 rows being used

$$a_0 = \frac{1704/2}{24}$$

$$a_1 = \frac{650.745}{24}$$

$$b_1 = \frac{646.776}{24}$$

$$a_2 = \frac{-231.56}{24}$$

$$b_2 = \frac{-210.724}{24}$$

$$a_3 = \frac{225.26}{24}$$

$$b_3 = \frac{-232.776}{24}$$

Hence the sum of squared residuals for different values of $\underline{k}$ are

$$k = 0, \quad s^2 = 22{,}152$$

$$k = 1, \quad s^2 = 4{,}615$$

$$k = 2, \quad s^2 = 2{,}573$$

$$k = 3, \quad s^2 = 386$$

Applying Pearson's χ^2 test as before we find

$k = 0$, $\chi^2 = \frac{22{,}152}{196} = 113$ for 23 degrees of freedom.

$k = 1$, $\chi^2 = \frac{4{,}615}{196} = 23.5$ for 21 degrees of freedom.

$k = 2$, $\chi^2 = \frac{2{,}573}{196} = 13.1$ for 19 degrees of freedom.

$k = 3$, $\chi^2 = \frac{386}{196} = 2.0$ for 17 degrees of freedom.

The /

The corresponding values of P from the χ^2 tables are:-

$k = 0, \quad P < 0.01$

$k = 1, \quad P = 0.32$

$k = 2, \quad P = 0.83$

$k = 3, \quad P > 0.99$

It is therefore quite useless to graduate beyond $k = 2$. The best fits are to be obtained by giving k the value 2 or 1. The following smoothed values y are derived by premultiplying u by C for the case $2n = 24$, $k = 2$. (p. 63)

34.3	39.1	49.0	62.1	78.0	94.0	107.6	116.8
120.7	118.0	110.7	100.2	88.4	77.5	68.6	61.5
58.5	52.8	54.0	54.2	46.5	41.6	36.9	33.9

$\sum u = 1704$ and $\sum y = 1704.9$, which are in good agreement.

- - - oOo - - -

APPENDIX I.

Graduating Matrices $C = P(P'P)^{-1}P'$

I is the unit matrix, and Q is the matrix composed entirely of units.

$2n = 4$

$k = 0, \quad C = \frac{1}{4}Q$

$k = 1,$

$$C = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 & 1 \\ 1 & 3 & 1 & -1 \\ -1 & 1 & 3 & 1 \\ 1 & -1 & 1 & 3 \end{bmatrix}$$

$k = 2, \quad C = I.$

$2n = 6$

$k = 0, \quad C = \frac{1}{6}Q$

$k = 1,$

$$C = \frac{1}{6}\begin{bmatrix} 3 & 2 & \cdot & -1 & \cdot & 2 \\ 2 & 3 & 2 & \cdot & -1 & \cdot \\ \cdot & 2 & 3 & 2 & \cdot & -1 \\ -1 & \cdot & 2 & 3 & 2 & \cdot \\ \cdot & -1 & \cdot & 2 & 3 & 2 \\ 2 & \cdot & -1 & \cdot & 2 & 3 \end{bmatrix}$$

$k = 2,$

$$C = \frac{1}{6}\begin{bmatrix} 5 & 1 & -1 & 1 & -1 & 1 \\ 1 & 5 & 1 & -1 & 1 & -1 \\ -1 & 1 & 5 & 1 & -1 & 1 \\ 1 & -1 & 1 & 5 & 1 & -1 \\ -1 & 1 & -1 & 1 & 5 & 1 \\ 1 & -1 & 1 & -1 & 1 & 5 \end{bmatrix}$$

$k = 3 \quad C = I$

$\underline{2n = 8}$

$k=0, \quad C = \frac{1}{8}Q$

$k=1,$

$$C = \frac{1}{8}\begin{bmatrix} 3 & 1+\sqrt{2} & 1 & 1-\sqrt{2} & -1 & 1-\sqrt{2} & 1 & 1+\sqrt{2} \\ 1+\sqrt{2} & 3 & 1+\sqrt{2} & 1 & 1-\sqrt{2} & -1 & 1-\sqrt{2} & 1 \\ 1 & 1+\sqrt{2} & 3 & 1+\sqrt{2} & 1 & 1-\sqrt{2} & -1 & 1-\sqrt{2} \\ 1-\sqrt{2} & 1 & 1+\sqrt{2} & 3 & 1+\sqrt{2} & 1 & 1-\sqrt{2} & -1 \\ -1 & 1-\sqrt{2} & 1 & 1+\sqrt{2} & 3 & 1+\sqrt{2} & 1 & 1-\sqrt{2} \\ 1-\sqrt{2} & -1 & 1-\sqrt{2} & 1 & 1+\sqrt{2} & 3 & 1+\sqrt{2} & 1 \\ 1 & 1-\sqrt{2} & -1 & 1-\sqrt{2} & 1 & 1+\sqrt{2} & 3 & 1+\sqrt{2} \\ 1+\sqrt{2} & 1 & 1-\sqrt{2} & -1 & 1-\sqrt{2} & 1 & 1+\sqrt{2} & 3 \end{bmatrix}$$

$k=2,$

$$C = \frac{1}{8}\begin{bmatrix} 5 & 1+\sqrt{2} & -1 & 1-\sqrt{2} & 1 & 1-\sqrt{2} & -1 & 1+\sqrt{2} \\ 1+\sqrt{2} & 5 & 1+\sqrt{2} & -1 & 1-\sqrt{2} & 1 & 1-\sqrt{2} & -1 \\ -1 & 1+\sqrt{2} & 5 & 1+\sqrt{2} & -1 & 1-\sqrt{2} & 1 & 1-\sqrt{2} \\ 1-\sqrt{2} & -1 & 1+\sqrt{2} & 5 & 1+\sqrt{2} & -1 & 1-\sqrt{2} & 1 \\ 1 & 1-\sqrt{2} & -1 & 1+\sqrt{2} & 5 & 1+\sqrt{2} & -1 & 1-\sqrt{2} \\ 1-\sqrt{2} & 1 & 1-\sqrt{2} & -1 & 1+\sqrt{2} & 5 & 1+\sqrt{2} & -1 \\ -1 & 1-\sqrt{2} & 1 & 1-\sqrt{2} & -1 & 1+\sqrt{2} & 5 & 1+\sqrt{2} \\ 1+\sqrt{2} & -1 & 1-\sqrt{2} & 1 & 1-\sqrt{2} & -1 & 1+\sqrt{2} & 5 \end{bmatrix}$$

$k=3,$

$$C = \frac{1}{8}\begin{bmatrix} 7 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 7 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 7 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 7 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 7 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 7 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & 7 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & 7 \end{bmatrix}$$

$k=4, \quad C = I.$

$2n = 10$

$k = 0, \quad C = \frac{1}{10} Q.$

$k = 1,$

$$C = \frac{1}{20}\begin{bmatrix} 6 & 3+\sqrt5 & 1+\sqrt5 & 3-\sqrt5 & 1-\sqrt5 & -1 & 1-\sqrt5 & 3-\sqrt5 & 1+\sqrt5 & 3+\sqrt5 \\ 3+\sqrt5 & 6 & 3+\sqrt5 & 1+\sqrt5 & 3-\sqrt5 & 1-\sqrt5 & -1 & 1-\sqrt5 & 3-\sqrt5 & 1+\sqrt5 \\ 1+\sqrt5 & 3+\sqrt5 & 6 & 3+\sqrt5 & 1+\sqrt5 & 3-\sqrt5 & 1-\sqrt5 & -1 & 1-\sqrt5 & 3-\sqrt5 \\ 3-\sqrt5 & 1+\sqrt5 & 3+\sqrt5 & 6 & 3+\sqrt5 & 1+\sqrt5 & 3-\sqrt5 & 1-\sqrt5 & -1 & 1-\sqrt5 \\ 1-\sqrt5 & 3-\sqrt5 & 1+\sqrt5 & 3+\sqrt5 & 6 & 3+\sqrt5 & 1+\sqrt5 & 3-\sqrt5 & 1-\sqrt5 & -1 \\ -1 & 1-\sqrt5 & 3-\sqrt5 & 1+\sqrt5 & 3+\sqrt5 & 6 & 3+\sqrt5 & 1+\sqrt5 & 3-\sqrt5 & 1-\sqrt5 \\ 1-\sqrt5 & -1 & 1-\sqrt5 & 3-\sqrt5 & 1+\sqrt5 & 3+\sqrt5 & 6 & 3+\sqrt5 & 1+\sqrt5 & 3-\sqrt5 \\ 3-\sqrt5 & 1-\sqrt5 & -1 & 1-\sqrt5 & 3-\sqrt5 & 1+\sqrt5 & 3+\sqrt5 & 6 & 3+\sqrt5 & 1+\sqrt5 \\ 1+\sqrt5 & 3-\sqrt5 & 1-\sqrt5 & -1 & 1-\sqrt5 & 3-\sqrt5 & 1+\sqrt5 & 3+\sqrt5 & 6 & 3+\sqrt5 \\ 3+\sqrt5 & 1+\sqrt5 & 3-\sqrt5 & 1-\sqrt5 & -1 & 1-\sqrt5 & 3-\sqrt5 & 1+\sqrt5 & 3+\sqrt5 & 6 \end{bmatrix}$$

$k = 2,$

$$C = \frac{1}{10}\begin{bmatrix} 5 & 1+\sqrt5 & \cdot & 1-\sqrt5 & \cdot & 1 & \cdot & 1-\sqrt5 & \cdot & 1+\sqrt5 \\ 1+\sqrt5 & 5 & 1+\sqrt5 & \cdot & 1-\sqrt5 & \cdot & 1 & \cdot & 1-\sqrt5 & \cdot \\ \cdot & 1+\sqrt5 & 5 & 1+\sqrt5 & \cdot & 1-\sqrt5 & \cdot & 1 & \cdot & 1-\sqrt5 \\ 1-\sqrt5 & \cdot & 1+\sqrt5 & 5 & 1+\sqrt5 & \cdot & 1-\sqrt5 & \cdot & 1 & \cdot \\ \cdot & 1-\sqrt5 & \cdot & 1+\sqrt5 & 5 & 1+\sqrt5 & \cdot & 1-\sqrt5 & \cdot & 1 \\ 1 & \cdot & 1-\sqrt5 & \cdot & 1+\sqrt5 & 5 & 1+\sqrt5 & \cdot & 1-\sqrt5 & \cdot \\ \cdot & 1 & \cdot & 1-\sqrt5 & \cdot & 1+\sqrt5 & 5 & 1+\sqrt5 & \cdot & 1-\sqrt5 \\ 1-\sqrt5 & \cdot & 1 & \cdot & 1-\sqrt5 & \cdot & 1+\sqrt5 & 5 & 1+\sqrt5 & \cdot \\ \cdot & 1-\sqrt5 & \cdot & 1 & \cdot & 1-\sqrt5 & \cdot & 1+\sqrt5 & 5 & 1+\sqrt5 \\ 1+\sqrt5 & \cdot & 1-\sqrt5 & \cdot & 1 & \cdot & 1-\sqrt5 & \cdot & 1+\sqrt5 & 5 \end{bmatrix}$$

$k = 3,$

$$C = \frac{1}{20}\begin{bmatrix}
14 & 3+\sqrt5 & -(1+\sqrt5) & 3-\sqrt5 & -(1-\sqrt5) & -1 & -(1-\sqrt5) & 3-\sqrt5 & -(1+\sqrt5) & 3+\sqrt5 \\
3+\sqrt5 & 14 & 3+\sqrt5 & -(1+\sqrt5) & 3-\sqrt5 & -(1-\sqrt5) & -1 & -(1-\sqrt5) & 3-\sqrt5 & -(1+\sqrt5) \\
-(1+\sqrt5) & 3+\sqrt5 & 14 & 3+\sqrt5 & -(1+\sqrt5) & 3-\sqrt5 & -(1-\sqrt5) & -1 & -(1-\sqrt5) & 3-\sqrt5 \\
3-\sqrt5 & -(1+\sqrt5) & 3+\sqrt5 & 14 & 3+\sqrt5 & -(1+\sqrt5) & 3-\sqrt5 & -(1-\sqrt5) & -1 & -(1-\sqrt5) \\
-(1-\sqrt5) & 3-\sqrt5 & -(1+\sqrt5) & 3+\sqrt5 & 14 & 3+\sqrt5 & -(1+\sqrt5) & 3-\sqrt5 & -(1-\sqrt5) & -1 \\
-1 & -(1-\sqrt5) & 3-\sqrt5 & -(1+\sqrt5) & 3+\sqrt5 & 14 & 3+\sqrt5 & -(1+\sqrt5) & 3-\sqrt5 & -(1-\sqrt5) \\
-(1-\sqrt5) & -1 & -(1-\sqrt5) & 3-\sqrt5 & -(1+\sqrt5) & 3+\sqrt5 & 14 & 3+\sqrt5 & -(1+\sqrt5) & 3-\sqrt5 \\
3-\sqrt5 & -(1-\sqrt5) & -1 & -(1-\sqrt5) & 3-\sqrt5 & -(1+\sqrt5) & 3+\sqrt5 & 14 & 3+\sqrt5 & -(1+\sqrt5) \\
-(1+\sqrt5) & 3-\sqrt5 & -(1-\sqrt5) & -1 & -(1-\sqrt5) & 3-\sqrt5 & -(1+\sqrt5) & 3+\sqrt5 & 14 & 3+\sqrt5 \\
3+\sqrt5 & -(1+\sqrt5) & 3-\sqrt5 & -(1-\sqrt5) & -1 & -(1-\sqrt5) & 3-\sqrt5 & -(1+\sqrt5) & 3+\sqrt5 & 14
\end{bmatrix}$$

$k = 4,$

$$C = \frac{1}{10}\begin{bmatrix}
9 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & 9 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & 9 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & 9 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & 9 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & 9 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & 9 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & 9 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 9 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 9
\end{bmatrix}$$

$k = 5, \qquad C = I.$

$\underline{2n = 12}$

$k = 0, \quad C = \frac{1}{12} Q$

$k = 1,$

$$C = \frac{1}{12}\begin{bmatrix}
3 & 1+\sqrt{3} & 2 & 1 & \cdot & 1-\sqrt{3} & -1 & 1-\sqrt{3} & \cdot & 1 & 2 & 1+\sqrt{3} \\
1+\sqrt{3} & 3 & 1+\sqrt{3} & 2 & 1 & \cdot & 1-\sqrt{3} & -1 & 1-\sqrt{3} & \cdot & 1 & 2 \\
2 & 1+\sqrt{3} & 3 & 1+\sqrt{3} & 2 & 1 & \cdot & 1-\sqrt{3} & -1 & 1-\sqrt{3} & \cdot & 1 \\
1 & 2 & 1+\sqrt{3} & 3 & 1+\sqrt{3} & 2 & 1 & \cdot & 1-\sqrt{3} & -1 & 1-\sqrt{3} & \cdot \\
\cdot & 1 & 2 & 1+\sqrt{3} & 3 & 1+\sqrt{3} & 2 & 1 & \cdot & 1-\sqrt{3} & -1 & 1-\sqrt{3} \\
1-\sqrt{3} & \cdot & 1 & 2 & 1+\sqrt{3} & 3 & 1+\sqrt{3} & 2 & 1 & \cdot & 1-\sqrt{3} & -1 \\
-1 & 1-\sqrt{3} & \cdot & 1 & 2 & 1+\sqrt{3} & 3 & 1+\sqrt{3} & 2 & 1 & \cdot & 1-\sqrt{3} \\
1-\sqrt{3} & -1 & 1-\sqrt{3} & \cdot & 1 & 2 & 1+\sqrt{3} & 3 & 1+\sqrt{3} & 2 & 1 & \cdot \\
\cdot & 1-\sqrt{3} & -1 & 1-\sqrt{3} & \cdot & 1 & 2 & 1+\sqrt{3} & 3 & 1+\sqrt{3} & 2 & 1 \\
1 & \cdot & 1-\sqrt{3} & -1 & 1-\sqrt{3} & \cdot & 1 & 2 & 1+\sqrt{3} & 3 & 1+\sqrt{3} & 2 \\
2 & 1 & \cdot & 1-\sqrt{3} & -1 & 1-\sqrt{3} & \cdot & 1 & 2 & 1+\sqrt{3} & 3 & 1+\sqrt{3} \\
1+\sqrt{3} & 2 & 1 & \cdot & 1-\sqrt{3} & -1 & 1-\sqrt{3} & \cdot & 1 & 2 & 1+\sqrt{3} & 3
\end{bmatrix}$$

$k = 2,$

$$C = \frac{1}{12}\begin{bmatrix}
5 & 2+\sqrt{3} & 1 & -1 & -1 & 2-\sqrt{3} & 1 & 2-\sqrt{3} & -1 & -1 & 1 & 2+\sqrt{3} \\
2+\sqrt{3} & 5 & 2+\sqrt{3} & 1 & -1 & -1 & 2-\sqrt{3} & 1 & 2-\sqrt{3} & -1 & -1 & 1 \\
1 & 2+\sqrt{3} & 5 & 2+\sqrt{3} & 1 & -1 & -1 & 2-\sqrt{3} & 1 & 2-\sqrt{3} & -1 & -1 \\
-1 & 1 & 2+\sqrt{3} & 5 & 2+\sqrt{3} & 1 & -1 & -1 & 2-\sqrt{3} & 1 & 2-\sqrt{3} & -1 \\
-1 & -1 & 1 & 2+\sqrt{3} & 5 & 2+\sqrt{3} & 1 & -1 & -1 & 2-\sqrt{3} & 1 & 2-\sqrt{3} \\
2-\sqrt{3} & -1 & -1 & 1 & 2+\sqrt{3} & 5 & 2+\sqrt{3} & 1 & -1 & -1 & 2-\sqrt{3} & 1 \\
1 & 2-\sqrt{3} & -1 & -1 & 1 & 2+\sqrt{3} & 5 & 2+\sqrt{3} & 1 & -1 & -1 & 2-\sqrt{3} \\
2-\sqrt{3} & 1 & 2-\sqrt{3} & -1 & -1 & 1 & 2+\sqrt{3} & 5 & 2+\sqrt{3} & 1 & -1 & -1 \\
-1 & 2-\sqrt{3} & 1 & 2-\sqrt{3} & -1 & -1 & 1 & 2+\sqrt{3} & 5 & 2+\sqrt{3} & 1 & -1 \\
-1 & -1 & 2-\sqrt{3} & 1 & 2-\sqrt{3} & -1 & -1 & 1 & 2+\sqrt{3} & 5 & 2+\sqrt{3} & 1 \\
1 & -1 & -1 & 2-\sqrt{3} & 1 & 2-\sqrt{3} & -1 & -1 & 1 & 2+\sqrt{3} & 5 & 2+\sqrt{3} \\
2+\sqrt{3} & 1 & -1 & -1 & 2-\sqrt{3} & 1 & 2-\sqrt{3} & -1 & -1 & 1 & 2+\sqrt{3} & 5
\end{bmatrix}$$

$k = 3,$

$$C = \frac{1}{12}\begin{bmatrix}
7 & 2+\sqrt{3} & -1 & -1 & 1 & 2-\sqrt{3} & -1 & 2-\sqrt{3} & 1 & -1 & -1 & 2+\sqrt{3} \\
2+\sqrt{3} & 7 & 2+\sqrt{3} & -1 & -1 & 1 & 2-\sqrt{3} & -1 & 2-\sqrt{3} & 1 & -1 & -1 \\
-1 & 2+\sqrt{3} & 7 & 2+\sqrt{3} & -1 & -1 & 1 & 2-\sqrt{3} & -1 & 2-\sqrt{3} & 1 & -1 \\
-1 & -1 & 2+\sqrt{3} & 7 & 2+\sqrt{3} & -1 & -1 & 1 & 2-\sqrt{3} & -1 & 2-\sqrt{3} & 1 \\
1 & -1 & -1 & 2+\sqrt{3} & 7 & 2+\sqrt{3} & -1 & -1 & 1 & 2-\sqrt{3} & -1 & 2-\sqrt{3} \\
2-\sqrt{3} & 1 & -1 & -1 & 2+\sqrt{3} & 7 & 2+\sqrt{3} & -1 & -1 & 1 & 2-\sqrt{3} & -1 \\
-1 & 2-\sqrt{3} & 1 & -1 & -1 & 2+\sqrt{3} & 7 & 2+\sqrt{3} & -1 & -1 & 1 & 2-\sqrt{3} \\
2-\sqrt{3} & -1 & 2-\sqrt{3} & 1 & -1 & -1 & 2+\sqrt{3} & 7 & 2+\sqrt{3} & -1 & -1 & 1 \\
1 & 2-\sqrt{3} & -1 & 2-\sqrt{3} & 1 & -1 & -1 & 2+\sqrt{3} & 7 & 2+\sqrt{3} & -1 & -1 \\
-1 & 1 & 2-\sqrt{3} & -1 & 2-\sqrt{3} & 1 & -1 & -1 & 2+\sqrt{3} & 7 & 2+\sqrt{3} & -1 \\
-1 & -1 & 1 & 2-\sqrt{3} & -1 & 2-\sqrt{3} & 1 & -1 & -1 & 2+\sqrt{3} & 7 & 2+\sqrt{3} \\
2+\sqrt{3} & -1 & -1 & 1 & 2-\sqrt{3} & -1 & 2-\sqrt{3} & 1 & -1 & -1 & 2+\sqrt{3} & 7
\end{bmatrix}$$

$k = 4,$

$$C = \frac{1}{12}\begin{bmatrix}
9 & 1+\sqrt{3} & -2 & 1 & \cdot & 1-\sqrt{3} & 1 & 1-\sqrt{3} & \cdot & 1 & -2 & 1+\sqrt{3} \\
1+\sqrt{3} & 9 & 1+\sqrt{3} & -2 & 1 & \cdot & 1-\sqrt{3} & 1 & 1-\sqrt{3} & \cdot & 1 & -2 \\
-2 & 1+\sqrt{3} & 9 & 1+\sqrt{3} & -2 & 1 & \cdot & 1-\sqrt{3} & 1 & 1-\sqrt{3} & \cdot & 1 \\
1 & -2 & 1+\sqrt{3} & 9 & 1+\sqrt{3} & -2 & 1 & \cdot & 1-\sqrt{3} & 1 & 1-\sqrt{3} & \cdot \\
\cdot & 1 & -2 & 1+\sqrt{3} & 9 & 1+\sqrt{3} & -2 & 1 & \cdot & 1-\sqrt{3} & 1 & 1-\sqrt{3} \\
1-\sqrt{3} & \cdot & 1 & -2 & 1+\sqrt{3} & 9 & 1+\sqrt{3} & -2 & 1 & \cdot & 1-\sqrt{3} & 1 \\
1 & 1-\sqrt{3} & \cdot & 1 & -2 & 1+\sqrt{3} & 9 & 1+\sqrt{3} & -2 & 1 & \cdot & 1-\sqrt{3} \\
1-\sqrt{3} & 1 & 1-\sqrt{3} & \cdot & 1 & -2 & 1+\sqrt{3} & 9 & 1+\sqrt{3} & -2 & 1 & \cdot \\
\cdot & 1-\sqrt{3} & 1 & 1-\sqrt{3} & \cdot & 1 & -2 & 1+\sqrt{3} & 9 & 1+\sqrt{3} & -2 & 1 \\
1 & \cdot & 1-\sqrt{3} & 1 & 1-\sqrt{3} & \cdot & 1 & -2 & 1+\sqrt{3} & 9 & 1+\sqrt{3} & -2 \\
-2 & 1 & \cdot & 1-\sqrt{3} & 1 & 1-\sqrt{3} & \cdot & 1 & -2 & 1+\sqrt{3} & 9 & 1+\sqrt{3} \\
1+\sqrt{3} & -2 & 1 & \cdot & 1-\sqrt{3} & 1 & 1-\sqrt{3} & \cdot & 1 & -2 & 1+\sqrt{3} & 9
\end{bmatrix}$$

$k = 5,$

$$C = \frac{1}{12}\begin{bmatrix} 11 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 11 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 11 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 11 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 11 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 11 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & 11 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & 11 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 11 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 11 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 11 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 11 \end{bmatrix}$$

$k = 6, \qquad C = I.$

$\underline{2n = 16}$

$k=0, \quad C = \frac{1}{16} Q$

$k=1,$

(Cols. 1-8)

$$C = \frac{1}{16}\left[\begin{array}{cccccccc}
3 & 1+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2+\sqrt{2}} \\
1+\sqrt{2+\sqrt{2}} & 3 & 1+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} \\
1+\sqrt{2} & 1+\sqrt{2+\sqrt{2}} & 3 & 1+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2-\sqrt{2}} \\
1+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2+\sqrt{2}} & 3 & 1+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2-\sqrt{2}} & 1 \\
1 & 1+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2+\sqrt{2}} & 3 & 1+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2-\sqrt{2}} \\
1-\sqrt{2-\sqrt{2}} & 1 & 1+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2+\sqrt{2}} & 3 & 1+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} \\
1-\sqrt{2} & 1-\sqrt{2-\sqrt{2}} & 1 & 1+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2+\sqrt{2}} & 3 & 1+\sqrt{2+\sqrt{2}} \\
1-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2-\sqrt{2}} & 1 & 1+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2+\sqrt{2}} & 3 \\
-1 & 1-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2-\sqrt{2}} & 1 & 1+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2+\sqrt{2}} \\
1-\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2-\sqrt{2}} & 1 & 1+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} \\
1-\sqrt{2} & 1-\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2-\sqrt{2}} & 1 & 1+\sqrt{2-\sqrt{2}} \\
1-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2-\sqrt{2}} & 1 \\
1 & 1-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2-\sqrt{2}} \\
1+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} \\
1+\sqrt{2} & 1+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2+\sqrt{2}} \\
1+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2+\sqrt{2}} & -1
\end{array}\right.$$

(Cols. 9-16)

$$\left.\begin{array}{cccccccc}
-1 & 1-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2-\sqrt{2}} & 1 & 1+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2+\sqrt{2}} \\
1-\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2-\sqrt{2}} & 1 & 1+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} \\
1-\sqrt{2} & 1-\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2-\sqrt{2}} & 1 & 1+\sqrt{2-\sqrt{2}} \\
1-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2-\sqrt{2}} & 1 \\
1 & 1-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2-\sqrt{2}} \\
1+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} \\
1+\sqrt{2} & 1+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2+\sqrt{2}} \\
1+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2+\sqrt{2}} & -1 \\
3 & 1+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2+\sqrt{2}} \\
1+\sqrt{2+\sqrt{2}} & 3 & 1+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} \\
1+\sqrt{2} & 1+\sqrt{2+\sqrt{2}} & 3 & 1+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2-\sqrt{2}} \\
1+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2+\sqrt{2}} & 3 & 1+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2-\sqrt{2}} & 1 \\
1 & 1+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2+\sqrt{2}} & 3 & 1+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2-\sqrt{2}} \\
1-\sqrt{2-\sqrt{2}} & 1 & 1+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2+\sqrt{2}} & 3 & 1+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} \\
1-\sqrt{2} & 1-\sqrt{2-\sqrt{2}} & 1 & 1+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2+\sqrt{2}} & 3 & 1+\sqrt{2+\sqrt{2}} \\
1-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2-\sqrt{2}} & 1 & 1+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2+\sqrt{2}} & 3
\end{array}\right]$$

$k=2,$

(Cols. 1-8)

$$C=\frac{1}{16}\begin{bmatrix}
5 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} \\
1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 5 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} \\
1+\sqrt{2} & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 5 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} \\
1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 5 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -1 \\
-1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 5 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} \\
1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 5 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} \\
1-\sqrt{2} & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 5 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} \\
1+\sqrt{2}-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 5 \\
-3 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} \\
1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -3 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} \\
1-\sqrt{2} & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -3 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} \\
1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -3 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -1 \\
-1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -3 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} \\
1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -3 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} \\
1+\sqrt{2} & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -3 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} \\
1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -3
\end{bmatrix}$$

(Cols. 9-16)

$$\begin{bmatrix}
-3 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} \\
1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -3 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} \\
1-\sqrt{2} & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -3 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} \\
1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -3 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -1 \\
-1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -3 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} \\
1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -3 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} \\
1+\sqrt{2} & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -3 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} \\
1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -3 \\
5 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} \\
1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 5 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1-\sqrt{2} \\
1+\sqrt{2} & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 5 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} \\
1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 5 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -1 \\
-1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 5 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} \\
1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 5 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 1+\sqrt{2} \\
1-\sqrt{2} & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 5 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} \\
1+\sqrt{2}-\sqrt{2+\sqrt{2}} & 1-\sqrt{2} & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1+\sqrt{2} & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 5
\end{bmatrix}$$

$k = 3,$

(Cols. 1-8)

$$C = \frac{1}{16}\left[\begin{array}{cccccccc}
y & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & 1 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} \\
1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & y & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & 1 \\
1 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & y & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} \\
1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & 1 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & y & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & -1 \\
-1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & 1 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & y & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} \\
1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & 1 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & y & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & 1 \\
1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & 1 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & y & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} \\
1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & 1 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & y \\
-5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & 1 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} \\
1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & -5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & 1 \\
1 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & -5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} \\
1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & 1 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & -5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & -1 \\
-1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & 1 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & -5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} \\
1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & 1 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & -5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & 1 \\
1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & 1 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & -5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} \\
1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & 1 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & -5
\end{array}\right.$$

(Cols. 9- 16)

$$\left.\begin{array}{cccccccc}
-5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & 1 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} \\
1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & -5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & 1 \\
1 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & -5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} \\
1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & 1 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & -5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & -1 \\
-1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & 1 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & -5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} \\
1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & 1 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & -5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & 1 \\
1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & 1 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & -5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} \\
1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & 1 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & -5 \\
y & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & 1 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} \\
1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & y & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & 1 \\
1 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & y & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} \\
1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & 1 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & y & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & -1 \\
-1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & 1 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & y & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} \\
1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & 1 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & y & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & 1 \\
1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & 1 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & y & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} \\
1+\sqrt{2}-\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}} & -1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}} & 1 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} & y
\end{array}\right]$$

$k=4,$

(Cols. 1–8)

$$C=\frac{1}{16}\left[\begin{matrix}
9 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & -1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & 1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & -1 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} \\
1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & 9 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & -1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & 1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & -1 \\
-1 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & 9 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & -1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & 1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} \\
1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & -1 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & 9 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & -1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & 1 \\
1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & -1 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & 9 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & -1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} \\
1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & 1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & -1 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & 9 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & -1 \\
-1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & 1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & -1 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & 9 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} \\
1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & 1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & -1 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & 9 \\
-3 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & 1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & -1 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} \\
1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -3 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & 1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & -1 \\
-1 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -3 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & 1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} \\
1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & -1 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -3 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & 1 \\
1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & -1 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -3 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} \\
1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & 1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & -1 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -3 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -1 \\
-1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & 1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & -1 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -3 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} \\
1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & -1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & 1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & -1 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -3
\end{matrix}\right.$$

(Cols. 9–16)

$$\left.\begin{matrix}
-3 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & 1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & -1 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} \\
1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -3 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & 1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & -1 \\
-1 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -3 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & 1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} \\
1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & -1 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -3 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & 1 \\
1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & -1 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -3 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} \\
1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & 1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & -1 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -3 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -1 \\
-1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & 1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & -1 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -3 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} \\
1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & -1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & 1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & -1 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -3 \\
9 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & -1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & 1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & -1 & 1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} \\
1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & 9 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & -1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & 1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & -1 \\
-1 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & 9 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & -1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & 1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} \\
1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & -1 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & 9 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & -1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & 1 \\
1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & -1 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & 9 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & -1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} \\
1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & 1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & -1 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & 9 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & -1 \\
-1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & 1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & -1 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & 9 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} \\
1+\sqrt2-\sqrt{2+\sqrt2}-\sqrt{2-\sqrt2} & -1 & 1-\sqrt2-\sqrt{2-\sqrt2}+\sqrt{2+\sqrt2} & 1 & 1-\sqrt2+\sqrt{2-\sqrt2}-\sqrt{2+\sqrt2} & -1 & 1+\sqrt2+\sqrt{2+\sqrt2}+\sqrt{2-\sqrt2} & 9
\end{matrix}\right]$$

$k=5,$

(Cols. 1-8)

$$C=\frac{1}{16}\begin{bmatrix}
11 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} \\
1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 11 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) \\
-(1+\sqrt{2}) & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 11 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} \\
1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 11 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1 \\
1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 11 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} \\
1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 11 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) \\
-(1-\sqrt{2}) & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 11 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} \\
1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 11 \\
-5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} \\
1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) \\
-(1-\sqrt{2}) & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} \\
1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1 \\
1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} \\
1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) \\
-(1+\sqrt{2}) & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} \\
1+\sqrt{2}+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -5
\end{bmatrix}$$

(Cols. 9-16)

$$\begin{bmatrix}
-5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} \\
1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) \\
-(1-\sqrt{2}) & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} \\
1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1 \\
1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} \\
1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) \\
-(1+\sqrt{2}) & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -5 & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} \\
1+\sqrt{2}+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -5 \\
11 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) & 1+\sqrt{2}-\sqrt{2+\sqrt{2}} \\
1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 11 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) \\
-(1+\sqrt{2}) & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 11 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} \\
1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 11 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & 1 \\
1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 11 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} \\
1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 11 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) \\
-(1-\sqrt{2}) & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 11 & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} \\
1+\sqrt{2}-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2}-\sqrt{2-\sqrt{2}} & 1 & 1-\sqrt{2}+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2}+\sqrt{2+\sqrt{2}} & 11
\end{bmatrix}$$

$k=6,$

(Cols. 1-8)

$$C=\frac{1}{16}\left[\begin{matrix}
13 & 1+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2+\sqrt{2}} \\
1+\sqrt{2+\sqrt{2}} & 13 & 1+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) \\
-(1+\sqrt{2}) & 1+\sqrt{2+\sqrt{2}} & 13 & 1+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2-\sqrt{2}} \\
1+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2+\sqrt{2}} & 13 & 1+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2-\sqrt{2}} & -1 \\
-1 & 1+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2+\sqrt{2}} & 13 & 1+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2-\sqrt{2}} \\
1-\sqrt{2-\sqrt{2}} & -1 & 1+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2+\sqrt{2}} & 13 & 1+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) \\
-(1-\sqrt{2}) & 1-\sqrt{2-\sqrt{2}} & -1 & 1+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2+\sqrt{2}} & 13 & 1+\sqrt{2+\sqrt{2}} \\
1-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2-\sqrt{2}} & -1 & 1+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2+\sqrt{2}} & 13 \\
-3 & 1-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2-\sqrt{2}} & -1 & 1+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2+\sqrt{2}} \\
1-\sqrt{2+\sqrt{2}} & -3 & 1-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2-\sqrt{2}} & -1 & 1+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) \\
-(1-\sqrt{2}) & 1-\sqrt{2+\sqrt{2}} & -3 & 1-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2-\sqrt{2}} & -1 & 1+\sqrt{2-\sqrt{2}} \\
1-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2+\sqrt{2}} & -3 & 1-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2-\sqrt{2}} & -1 \\
-1 & 1-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2+\sqrt{2}} & -3 & 1-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2-\sqrt{2}} \\
1+\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2+\sqrt{2}} & -3 & 1-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) \\
-(1+\sqrt{2}) & 1+\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2+\sqrt{2}} & -3 & 1-\sqrt{2+\sqrt{2}} \\
1+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2+\sqrt{2}} & -3
\end{matrix}\right.$$

(Cols. 9-16)

$$\left.\begin{matrix}
-3 & 1-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2-\sqrt{2}} & -1 & 1+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2+\sqrt{2}} \\
1-\sqrt{2+\sqrt{2}} & -3 & 1-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2-\sqrt{2}} & -1 & 1+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) \\
-(1-\sqrt{2}) & 1-\sqrt{2+\sqrt{2}} & -3 & 1-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2-\sqrt{2}} & -1 & 1+\sqrt{2-\sqrt{2}} \\
1-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2+\sqrt{2}} & -3 & 1-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2-\sqrt{2}} & -1 \\
-1 & 1-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2+\sqrt{2}} & -3 & 1-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2-\sqrt{2}} \\
1+\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2+\sqrt{2}} & -3 & 1-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) \\
-(1+\sqrt{2}) & 1+\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2+\sqrt{2}} & -3 & 1-\sqrt{2+\sqrt{2}} \\
1+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2+\sqrt{2}} & -3 \\
13 & 1+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2+\sqrt{2}} \\
1+\sqrt{2+\sqrt{2}} & 13 & 1+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2-\sqrt{2}} & -(1-\sqrt{2}) \\
-(1+\sqrt{2}) & 1+\sqrt{2+\sqrt{2}} & 13 & 1+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2-\sqrt{2}} & -1 & 1-\sqrt{2-\sqrt{2}} \\
1+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2+\sqrt{2}} & 13 & 1+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2-\sqrt{2}} & -1 \\
-1 & 1+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2+\sqrt{2}} & 13 & 1+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2-\sqrt{2}} \\
1-\sqrt{2-\sqrt{2}} & -1 & 1+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2+\sqrt{2}} & 13 & 1+\sqrt{2+\sqrt{2}} & -(1+\sqrt{2}) \\
-(1-\sqrt{2}) & 1-\sqrt{2-\sqrt{2}} & -1 & 1+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2+\sqrt{2}} & 13 & 1+\sqrt{2+\sqrt{2}} \\
1-\sqrt{2+\sqrt{2}} & -(1-\sqrt{2}) & 1-\sqrt{2-\sqrt{2}} & -1 & 1+\sqrt{2-\sqrt{2}} & -(1+\sqrt{2}) & 1+\sqrt{2+\sqrt{2}} & 13
\end{matrix}\right]$$

$k = 7,$

$$C = \frac{1}{16}\begin{bmatrix}
15 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & 15 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & 15 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & 15 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & 15 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & 15 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & 15 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & 15 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 15 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 15 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 15 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 15 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 15 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 15 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 15 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 15
\end{bmatrix}$$

$k = 8, \qquad C = I$

$\underline{2n = 24}$

$k=0, \quad C = \frac{1}{24} Q.$

$k=1, \quad C = \frac{1}{24}$ of

((Cols. 1– 12))

$$
\left[\begin{array}{cccccccccccc}
6 & 2+\sqrt6+\sqrt2 & 2+2\sqrt3 & 2+2\sqrt2 & 4 & 2+\sqrt6-\sqrt2 & 2 & 2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 & 2-2\sqrt3 & 2-\sqrt6-\sqrt2 \\
2+\sqrt6+\sqrt2 & 6 & 2+\sqrt6+\sqrt2 & 2+2\sqrt3 & 2+2\sqrt2 & 4 & 2+\sqrt6-\sqrt2 & 2 & 2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 & 2-2\sqrt3 \\
2+2\sqrt3 & 2+\sqrt6+\sqrt2 & 6 & 2+\sqrt6+\sqrt2 & 2+2\sqrt3 & 2+2\sqrt2 & 4 & 2+\sqrt6-\sqrt2 & 2 & 2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 \\
2+2\sqrt2 & 2+2\sqrt3 & 2+\sqrt6+\sqrt2 & 6 & 2+\sqrt6+\sqrt2 & 2+2\sqrt3 & 2+2\sqrt2 & 4 & 2+\sqrt6-\sqrt2 & 2 & 2-\sqrt6+\sqrt2 & \cdot \\
4 & 2+2\sqrt2 & 2+2\sqrt3 & 2+\sqrt6+\sqrt2 & 6 & 2+\sqrt6+\sqrt2 & 2+2\sqrt3 & 2+2\sqrt2 & 4 & 2+\sqrt6-\sqrt2 & 2 & 2-\sqrt6+\sqrt2 \\
2+\sqrt6-\sqrt2 & 4 & 2+2\sqrt2 & 2+2\sqrt3 & 2+\sqrt6+\sqrt2 & 6 & 2+\sqrt6+\sqrt2 & 2+2\sqrt3 & 2+2\sqrt2 & 4 & 2+\sqrt6-\sqrt2 & 2 \\
2 & 2+\sqrt6-\sqrt2 & 4 & 2+2\sqrt2 & 2+2\sqrt3 & 2+\sqrt6+\sqrt2 & 6 & 2+\sqrt6+\sqrt2 & 2+2\sqrt3 & 2+2\sqrt2 & 4 & 2+\sqrt6-\sqrt2 \\
2-\sqrt6+\sqrt2 & 2 & 2+\sqrt6-\sqrt2 & 4 & 2+2\sqrt2 & 2+2\sqrt3 & 2+\sqrt6+\sqrt2 & 6 & 2+\sqrt6+\sqrt2 & 2+2\sqrt3 & 2+2\sqrt2 & 4 \\
\cdot & 2-\sqrt6+\sqrt2 & 2 & 2+\sqrt6-\sqrt2 & 4 & 2+2\sqrt2 & 2+2\sqrt3 & 2+\sqrt6+\sqrt2 & 6 & 2+\sqrt6+\sqrt2 & 2+2\sqrt3 & 2+2\sqrt2 \\
2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 & 2 & 2+\sqrt6-\sqrt2 & 4 & 2+2\sqrt2 & 2+2\sqrt3 & 2+\sqrt6+\sqrt2 & 6 & 2+\sqrt6+\sqrt2 & 2+2\sqrt3 \\
2-2\sqrt3 & 2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 & 2 & 2+\sqrt6-\sqrt2 & 4 & 2+2\sqrt2 & 2+2\sqrt3 & 2+\sqrt6+\sqrt2 & 6 & 2+\sqrt6+\sqrt2 \\
2-\sqrt6-\sqrt2 & 2-2\sqrt3 & 2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 & 2 & 2+\sqrt6-\sqrt2 & 4 & 2+2\sqrt2 & 2+2\sqrt3 & 2+\sqrt6+\sqrt2 & 6 \\
-2 & 2-\sqrt6-\sqrt2 & 2-2\sqrt3 & 2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 & 2 & 2+\sqrt6-\sqrt2 & 4 & 2+2\sqrt2 & 2+2\sqrt3 & 2+\sqrt6+\sqrt2 \\
2-\sqrt6-\sqrt3 & -2 & 2-\sqrt6-\sqrt2 & 2-2\sqrt3 & 2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 & 2 & 2+\sqrt6-\sqrt2 & 4 & 2+2\sqrt2 & 2+2\sqrt3 \\
2-2\sqrt3 & 2-\sqrt6-\sqrt2 & -2 & 2-\sqrt6-\sqrt2 & 2-2\sqrt3 & 2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 & 2 & 2+\sqrt6-\sqrt2 & 4 & 2+2\sqrt2 \\
2-2\sqrt2 & 2-2\sqrt3 & 2-\sqrt6-\sqrt2 & -2 & 2-\sqrt6-\sqrt2 & 2-2\sqrt3 & 2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 & 2 & 2+\sqrt6-\sqrt2 & 4 \\
\cdot & 2-2\sqrt2 & 2-2\sqrt3 & 2-\sqrt6-\sqrt2 & -2 & 2-\sqrt6-\sqrt2 & 2-2\sqrt3 & 2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 & 2 & 2+\sqrt6-\sqrt2 \\
2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 & 2-2\sqrt3 & 2-\sqrt6-\sqrt2 & -2 & 2-\sqrt6-\sqrt2 & 2-2\sqrt3 & 2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 & 2 \\
2 & 2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 & 2-2\sqrt3 & 2-\sqrt6-\sqrt2 & -2 & 2-\sqrt6-\sqrt2 & 2-2\sqrt3 & 2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 \\
2+\sqrt6-\sqrt2 & 2 & 2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 & 2-2\sqrt3 & 2-\sqrt6-\sqrt2 & -2 & 2-\sqrt6-\sqrt2 & 2-2\sqrt3 & 2-2\sqrt2 & \cdot \\
4 & 2+\sqrt6-\sqrt2 & 2 & 2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 & 2-2\sqrt3 & 2-\sqrt6-\sqrt2 & -2 & 2-\sqrt6-\sqrt2 & 2-2\sqrt3 & 2-2\sqrt2 \\
2+2\sqrt2 & 4 & 2+\sqrt6-\sqrt2 & 2 & 2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 & 2-2\sqrt3 & 2-\sqrt6-\sqrt2 & -2 & 2-\sqrt6-\sqrt2 & 2-2\sqrt3 \\
2+2\sqrt3 & 2+2\sqrt2 & 4 & 2+\sqrt6-\sqrt2 & 2 & 2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 & 2-2\sqrt3 & 2-\sqrt6-\sqrt2 & -2 & 2-\sqrt6-\sqrt2 \\
2+\sqrt6+\sqrt2 & 2+2\sqrt3 & 2+2\sqrt2 & 4 & 2+\sqrt6-\sqrt2 & 2 & 2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 & 2-2\sqrt3 & 2-\sqrt6-\sqrt2 & -2
\end{array}\right.
$$

(Cols. 13-24)

$$
\begin{bmatrix}
-2 & 2-\sqrt6-\sqrt2 & 2-2\sqrt3 & 2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 & 2 & 2+\sqrt6-\sqrt2 & 4 & 2+2\sqrt2 & 2+2\sqrt3 & 2+\sqrt6+\sqrt2 \\
2-\sqrt6-\sqrt2 & -2 & 2-\sqrt6-\sqrt2 & 2-2\sqrt3 & 2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 & 2 & 2+\sqrt6-\sqrt2 & 4 & 2+2\sqrt2 & 2+2\sqrt3 \\
2-2\sqrt3 & 2-\sqrt6-\sqrt2 & -2 & 2-\sqrt6-\sqrt2 & 2-2\sqrt3 & 2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 & 2 & 2+\sqrt6-\sqrt2 & 4 & 2+2\sqrt2 \\
2-2\sqrt2 & 2-2\sqrt3 & 2-\sqrt6-\sqrt2 & -2 & 2-\sqrt6-\sqrt2 & 2-2\sqrt3 & 2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 & 2 & 2+\sqrt6-\sqrt2 & 4 \\
\cdot & 2-2\sqrt2 & 2-2\sqrt3 & 2-\sqrt6-\sqrt2 & -2 & 2-\sqrt6-\sqrt2 & 2-2\sqrt3 & 2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 & 2 & 2+\sqrt6-\sqrt2 \\
2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 & 2-2\sqrt3 & 2-\sqrt6-\sqrt2 & -2 & 2-\sqrt6-\sqrt2 & 2-2\sqrt3 & 2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 & 2 \\
2 & 2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 & 2-2\sqrt3 & 2-\sqrt6-\sqrt2 & -2 & 2-\sqrt6-\sqrt2 & 2-2\sqrt3 & 2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 \\
2+\sqrt6-\sqrt2 & 2 & 2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 & 2-2\sqrt3 & 2-\sqrt6-\sqrt2 & -2 & 2-\sqrt6-\sqrt2 & 2-2\sqrt3 & 2-2\sqrt2 & \cdot \\
4 & 2+\sqrt6-\sqrt2 & 2 & 2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 & 2-2\sqrt3 & 2-\sqrt6-\sqrt2 & -2 & 2-\sqrt6-\sqrt2 & 2-2\sqrt3 & 2-2\sqrt2 \\
2+2\sqrt2 & 4 & 2+\sqrt6-\sqrt2 & 2 & 2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 & 2-2\sqrt3 & 2-\sqrt6-\sqrt2 & -2 & 2-\sqrt6-\sqrt2 & 2-2\sqrt3 \\
2+2\sqrt3 & 2+2\sqrt2 & 4 & 2+\sqrt6-\sqrt2 & 2 & 2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 & 2-2\sqrt3 & 2-\sqrt6-\sqrt2 & -2 & 2-\sqrt6-\sqrt2 \\
2+\sqrt6+\sqrt2 & 2+2\sqrt3 & 2+2\sqrt2 & 4 & 2+\sqrt6-\sqrt2 & 2 & 2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 & 2-2\sqrt3 & 2-\sqrt6-\sqrt2 & -2 \\
6 & 2+\sqrt6+\sqrt2 & 2+2\sqrt3 & 2+2\sqrt2 & 4 & 2+\sqrt6-\sqrt2 & 2 & 2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 & 2-2\sqrt3 & 2-\sqrt6-\sqrt2 \\
2+\sqrt6+\sqrt2 & 6 & 2+\sqrt6+\sqrt2 & 2+2\sqrt3 & 2+2\sqrt2 & 4 & 2+\sqrt6-\sqrt2 & 2 & 2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 & 2-2\sqrt3 \\
2+2\sqrt3 & 2+\sqrt6+\sqrt2 & 6 & 2+\sqrt6+\sqrt2 & 2+2\sqrt3 & 2+2\sqrt2 & 4 & 2+\sqrt6-\sqrt2 & 2 & 2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 \\
2+2\sqrt2 & 2+2\sqrt3 & 2+\sqrt6+\sqrt2 & 6 & 2+\sqrt6+\sqrt2 & 2+2\sqrt3 & 2+2\sqrt2 & 4 & 2+\sqrt6-\sqrt2 & 2 & 2-\sqrt6+\sqrt2 & \cdot \\
4 & 2+2\sqrt2 & 2+2\sqrt3 & 2+\sqrt6+\sqrt2 & 6 & 2+\sqrt6+\sqrt2 & 2+2\sqrt3 & 2+2\sqrt2 & 4 & 2+\sqrt6-\sqrt2 & 2 & 2-\sqrt6+\sqrt2 \\
2+\sqrt6-\sqrt2 & 4 & 2+2\sqrt2 & 2+2\sqrt3 & 2+\sqrt6+\sqrt2 & 6 & 2+\sqrt6+\sqrt2 & 2+2\sqrt3 & 2+2\sqrt2 & 4 & 2+\sqrt6-\sqrt2 & 2 \\
2 & 2+\sqrt6-\sqrt2 & 4 & 2+2\sqrt2 & 2+2\sqrt3 & 2+\sqrt6+\sqrt2 & 6 & 2+\sqrt6+\sqrt2 & 2+2\sqrt3 & 2+2\sqrt2 & 4 & 2+\sqrt6-\sqrt2 \\
2-\sqrt6+\sqrt2 & 2 & 2+\sqrt6-\sqrt2 & 4 & 2+2\sqrt2 & 2+2\sqrt3 & 2+\sqrt6+\sqrt2 & 6 & 2+\sqrt6+\sqrt2 & 2+2\sqrt3 & 2+2\sqrt2 & 4 \\
\cdot & 2-\sqrt6+\sqrt2 & 2 & 2+\sqrt6-\sqrt2 & 4 & 2+2\sqrt2 & 2+2\sqrt3 & 2+\sqrt6+\sqrt2 & 6 & 2+\sqrt6+\sqrt2 & 2+2\sqrt3 & 2+2\sqrt2 \\
2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 & 2 & 2+\sqrt6-\sqrt2 & 4 & 2+2\sqrt2 & 2+2\sqrt3 & 2+\sqrt6+\sqrt2 & 6 & 2+\sqrt6+\sqrt2 & 2+2\sqrt3 \\
2-2\sqrt3 & 2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 & 2 & 2+\sqrt6-\sqrt2 & 4 & 2+2\sqrt2 & 2+2\sqrt3 & 2+\sqrt6+\sqrt2 & 6 & 2+\sqrt6+\sqrt2 \\
2-\sqrt6-\sqrt2 & 2-2\sqrt3 & 2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 & 2 & 2+\sqrt6-\sqrt2 & 4 & 2+2\sqrt2 & 2+2\sqrt3 & 2+\sqrt6+\sqrt2 & 6
\end{bmatrix}
$$

$k = 2,$

$C = \frac{1}{48}$ of

(Cols. 1 - 12)

$$\left[\begin{array}{cccccccccccc}
10 & 2+\sqrt{6}+\sqrt{2}+2\sqrt{3} & 4+2\sqrt{3} & 2+2\sqrt{2} & 2 & 2+\sqrt{6}-\sqrt{2}-2\sqrt{3} & -2 & 2-\sqrt{6}+\sqrt{2}-2\sqrt{3} & -2 & 2-2\sqrt{2} & 4-2\sqrt{3} & 2-\sqrt{6}-\sqrt{2}+2\sqrt{3} \\
2+\sqrt{6}+\sqrt{2}+2\sqrt{3} & 10 & 2+\sqrt{6}+\sqrt{2}+2\sqrt{3} & 4+2\sqrt{3} & 2+2\sqrt{2} & 2 & 2+\sqrt{6}-\sqrt{2}-2\sqrt{3} & -2 & 2-\sqrt{6}+\sqrt{2}-2\sqrt{3} & -2 & 2-2\sqrt{2} & 4-2\sqrt{3} \\
4+2\sqrt{3} & 2+\sqrt{6}+\sqrt{2}+2\sqrt{3} & 10 & 2+\sqrt{6}+\sqrt{2}+2\sqrt{3} & 4+2\sqrt{3} & 2+2\sqrt{2} & 2 & 2+\sqrt{6}-\sqrt{2}-2\sqrt{3} & -2 & 2-\sqrt{6}+\sqrt{2}-2\sqrt{3} & -2 & 2-2\sqrt{2} \\
2+2\sqrt{2} & 4+2\sqrt{3} & 2+\sqrt{6}+\sqrt{2}+2\sqrt{3} & 10 & 2+\sqrt{6}+\sqrt{2}+2\sqrt{3} & 4+2\sqrt{3} & 2+2\sqrt{2} & 2 & 2+\sqrt{6}-\sqrt{2}-2\sqrt{3} & -2 & 2-\sqrt{6}+\sqrt{2}-2\sqrt{3} & -2 \\
2 & 2+2\sqrt{2} & 4+2\sqrt{3} & 2+\sqrt{6}+\sqrt{2}+2\sqrt{3} & 10 & 2+\sqrt{6}+\sqrt{2}+2\sqrt{3} & 4+2\sqrt{3} & 2+2\sqrt{2} & 2 & 2+\sqrt{6}-\sqrt{2}-2\sqrt{3} & -2 & 2-\sqrt{6}+\sqrt{2}-2\sqrt{3} \\
2+\sqrt{6}-\sqrt{2}-2\sqrt{3} & 2 & 2+2\sqrt{2} & 4+2\sqrt{3} & 2+\sqrt{6}+\sqrt{2}+2\sqrt{3} & 10 & 2+\sqrt{6}+\sqrt{2}+2\sqrt{3} & 4+2\sqrt{3} & 2+2\sqrt{2} & 2 & 2+\sqrt{6}-\sqrt{2}-2\sqrt{3} & -2 \\
-2 & 2+\sqrt{6}-\sqrt{2}-2\sqrt{3} & 2 & 2+2\sqrt{2} & 4+2\sqrt{3} & 2+\sqrt{6}+\sqrt{2}+2\sqrt{3} & 10 & 2+\sqrt{6}+\sqrt{2}+2\sqrt{3} & 4+2\sqrt{3} & 2+2\sqrt{2} & 2 & 2+\sqrt{6}-\sqrt{2}-2\sqrt{3} \\
2-\sqrt{6}+\sqrt{2}-2\sqrt{3} & -2 & 2+\sqrt{6}-\sqrt{2}-2\sqrt{3} & 2 & 2+2\sqrt{2} & 4+2\sqrt{3} & 2+\sqrt{6}+\sqrt{2}+2\sqrt{3} & 10 & 2+\sqrt{6}+\sqrt{2}+2\sqrt{3} & 4+2\sqrt{3} & 2+2\sqrt{2} & 2 \\
-2 & 2-\sqrt{6}+\sqrt{2}-2\sqrt{3} & -2 & 2+\sqrt{6}-\sqrt{2}-2\sqrt{3} & 2 & 2+2\sqrt{2} & 4+2\sqrt{3} & 2+\sqrt{6}+\sqrt{2}+2\sqrt{3} & 10 & 2+\sqrt{6}+\sqrt{2}+2\sqrt{3} & 4+2\sqrt{3} & 2+2\sqrt{2} \\
2-2\sqrt{2} & -2 & 2-\sqrt{6}+\sqrt{2}-2\sqrt{3} & -2 & 2+\sqrt{6}-\sqrt{2}-2\sqrt{3} & 2 & 2+2\sqrt{2} & 4+2\sqrt{3} & 2+\sqrt{6}+\sqrt{2}+2\sqrt{3} & 10 & 2+\sqrt{6}+\sqrt{2}+2\sqrt{3} & 4+2\sqrt{3} \\
4-2\sqrt{3} & 2-2\sqrt{2} & -2 & 2-\sqrt{6}+\sqrt{2}-2\sqrt{3} & -2 & 2+\sqrt{6}-\sqrt{2}-2\sqrt{3} & 2 & 2+2\sqrt{2} & 4+2\sqrt{3} & 2+\sqrt{6}+\sqrt{2}+2\sqrt{3} & 10 & 2+\sqrt{6}+\sqrt{2}+2\sqrt{3} \\
2-\sqrt{6}-\sqrt{2}+2\sqrt{3} & 4-2\sqrt{3} & 2-2\sqrt{2} & -2 & 2-\sqrt{6}+\sqrt{2}-2\sqrt{3} & -2 & 2+\sqrt{6}-\sqrt{2}-2\sqrt{3} & 2 & 2+2\sqrt{2} & 4+2\sqrt{3} & 2+\sqrt{6}+\sqrt{2}+2\sqrt{3} & 10 \\
2 & 2-\sqrt{6}-\sqrt{2}+2\sqrt{3} & 4-2\sqrt{3} & 2-2\sqrt{2} & -2 & 2-\sqrt{6}+\sqrt{2}-2\sqrt{3} & -2 & 2+\sqrt{6}-\sqrt{2}-2\sqrt{3} & 2 & 2+2\sqrt{2} & 4+2\sqrt{3} & 2+\sqrt{6}+\sqrt{2}+2\sqrt{3} \\
2-\sqrt{6}-\sqrt{2}+2\sqrt{3} & 2 & 2-\sqrt{6}-\sqrt{2}+2\sqrt{3} & 4-2\sqrt{3} & 2-2\sqrt{2} & -2 & 2-\sqrt{6}+\sqrt{2}-2\sqrt{3} & -2 & 2+\sqrt{6}-\sqrt{2}-2\sqrt{3} & 2 & 2+2\sqrt{2} & 4+2\sqrt{3} \\
4-2\sqrt{3} & 2-\sqrt{6}-\sqrt{2}+2\sqrt{3} & 2 & 2-\sqrt{6}-\sqrt{2}+2\sqrt{3} & 4-2\sqrt{3} & 2-2\sqrt{2} & -2 & 2-\sqrt{6}+\sqrt{2}-2\sqrt{3} & -2 & 2+\sqrt{6}-\sqrt{2}-2\sqrt{3} & 2 & 2+2\sqrt{2} \\
2-2\sqrt{2} & 4-2\sqrt{3} & 2-\sqrt{6}-\sqrt{2}+2\sqrt{3} & 2 & 2-\sqrt{6}-\sqrt{2}+2\sqrt{3} & 4-2\sqrt{3} & 2-2\sqrt{2} & -2 & 2-\sqrt{6}+\sqrt{2}-2\sqrt{3} & -2 & 2+\sqrt{6}-\sqrt{2}-2\sqrt{3} & 2 \\
-2 & 2-2\sqrt{2} & 4-2\sqrt{3} & 2-\sqrt{6}-\sqrt{2}+2\sqrt{3} & 2 & 2-\sqrt{6}-\sqrt{2}+2\sqrt{3} & 4-2\sqrt{3} & 2-2\sqrt{2} & -2 & 2-\sqrt{6}+\sqrt{2}-2\sqrt{3} & -2 & 2+\sqrt{6}-\sqrt{2}-2\sqrt{3} \\
2-\sqrt{6}+\sqrt{2}-2\sqrt{3} & -2 & 2-2\sqrt{2} & 4-2\sqrt{3} & 2-\sqrt{6}-\sqrt{2}+2\sqrt{3} & 2 & 2-\sqrt{6}-\sqrt{2}+2\sqrt{3} & 4-2\sqrt{3} & 2-2\sqrt{2} & -2 & 2-\sqrt{6}+\sqrt{2}-2\sqrt{3} & -2 \\
-2 & 2-\sqrt{6}+\sqrt{2}-2\sqrt{3} & -2 & 2-2\sqrt{2} & 4-2\sqrt{3} & 2-\sqrt{6}-\sqrt{2}+2\sqrt{3} & 2 & 2-\sqrt{6}-\sqrt{2}+2\sqrt{3} & 4-2\sqrt{3} & 2-2\sqrt{2} & -2 & 2-\sqrt{6}+\sqrt{2}-2\sqrt{3} \\
2+\sqrt{6}-\sqrt{2}-2\sqrt{3} & -2 & 2-\sqrt{6}+\sqrt{2}-2\sqrt{3} & -2 & 2-2\sqrt{2} & 4-2\sqrt{3} & 2-\sqrt{6}-\sqrt{2}+2\sqrt{3} & 2 & 2-\sqrt{6}-\sqrt{2}+2\sqrt{3} & 4-2\sqrt{3} & 2-2\sqrt{2} & -2 \\
2 & 2+\sqrt{6}-\sqrt{2}-2\sqrt{3} & -2 & 2-\sqrt{6}+\sqrt{2}-2\sqrt{3} & -2 & 2-2\sqrt{2} & 4-2\sqrt{3} & 2-\sqrt{6}-\sqrt{2}+2\sqrt{3} & 2 & 2-\sqrt{6}-\sqrt{2}+2\sqrt{3} & 4-2\sqrt{3} & 2-2\sqrt{2} \\
2+2\sqrt{2} & 2 & 2+\sqrt{6}-\sqrt{2}-2\sqrt{3} & -2 & 2-\sqrt{6}+\sqrt{2}-2\sqrt{3} & -2 & 2-2\sqrt{2} & 4-2\sqrt{3} & 2-\sqrt{6}-\sqrt{2}+2\sqrt{3} & 2 & 2-\sqrt{6}-\sqrt{2}+2\sqrt{3} & 4-2\sqrt{3} \\
4+2\sqrt{3} & 2+2\sqrt{2} & 2 & 2+\sqrt{6}-\sqrt{2}-2\sqrt{3} & -2 & 2-\sqrt{6}+\sqrt{2}-2\sqrt{3} & -2 & 2-2\sqrt{2} & 4-2\sqrt{3} & 2-\sqrt{6}-\sqrt{2}+2\sqrt{3} & 2 & 2-\sqrt{6}-\sqrt{2}+2\sqrt{3} \\
2+\sqrt{6}+\sqrt{2}+2\sqrt{3} & 4+2\sqrt{3} & 2+2\sqrt{2} & 2 & 2+\sqrt{6}-\sqrt{2}-2\sqrt{3} & -2 & 2-\sqrt{6}+\sqrt{2}-2\sqrt{3} & -2 & 2-2\sqrt{2} & 4-2\sqrt{3} & 2-\sqrt{6}-\sqrt{2}+2\sqrt{3} & 2
\end{array}\right.$$

(Cols. 13– 24)

$$
\begin{bmatrix}
2 & 2-\sqrt6-\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2-2\sqrt2 & -2 & 2-\sqrt6+\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2+2\sqrt2 & 4+2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 \\
2-\sqrt6-\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2-2\sqrt2 & -2 & 2-\sqrt6+\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2+2\sqrt2 & 4+2\sqrt3 \\
4-2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2-2\sqrt2 & -2 & 2-\sqrt6+\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2+2\sqrt2 \\
2-2\sqrt2 & 4-2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2-2\sqrt2 & -2 & 2-\sqrt6+\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 \\
-2 & 2-2\sqrt2 & 4-2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2-2\sqrt2 & -2 & 2-\sqrt6+\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 \\
2-\sqrt6+\sqrt2-2\sqrt3 & -2 & 2-2\sqrt2 & 4-2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2-2\sqrt2 & -2 & 2-\sqrt6+\sqrt2-2\sqrt3 & -2 \\
-2 & 2-\sqrt6+\sqrt2-2\sqrt3 & -2 & 2-2\sqrt2 & 4-2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2-2\sqrt2 & -2 & 2-\sqrt6+\sqrt2-2\sqrt3 \\
2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+\sqrt2-2\sqrt3 & -2 & 2-2\sqrt2 & 4-2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2-2\sqrt2 & -2 \\
2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+\sqrt2-2\sqrt3 & -2 & 2-2\sqrt2 & 4-2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2-2\sqrt2 \\
2+2\sqrt2 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+\sqrt2-2\sqrt3 & -2 & 2-2\sqrt2 & 4-2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-\sqrt2+2\sqrt3 & 4-2\sqrt3 \\
4+2\sqrt3 & 2+2\sqrt2 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+\sqrt2-2\sqrt3 & -2 & 2-2\sqrt2 & 4-2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-\sqrt2+2\sqrt3 \\
2+\sqrt6+\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2+2\sqrt2 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+\sqrt2-2\sqrt3 & -2 & 2-2\sqrt2 & 4-2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & 2 \\
10 & 2+\sqrt6+\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2+2\sqrt2 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+\sqrt2-2\sqrt3 & -2 & 2-2\sqrt2 & 4-2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 \\
2+\sqrt6+\sqrt2+2\sqrt3 & 10 & 2+\sqrt6+\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2+2\sqrt2 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+\sqrt2-2\sqrt3 & -2 & 2-2\sqrt2 & 4-2\sqrt3 \\
4+2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 10 & 2+\sqrt6+\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2+2\sqrt2 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+\sqrt2-2\sqrt3 & -2 & 2-2\sqrt2 \\
2+2\sqrt2 & 4+2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 10 & 2+\sqrt6+\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2+2\sqrt2 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+\sqrt2-2\sqrt3 & -2 \\
2 & 2+2\sqrt2 & 4+2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 10 & 2+\sqrt6+\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2+2\sqrt2 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+\sqrt2-2\sqrt3 \\
2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2+2\sqrt2 & 4+2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 10 & 2+\sqrt6+\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2+2\sqrt2 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 \\
-2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2+2\sqrt2 & 4+2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 10 & 2+\sqrt6+\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2+2\sqrt2 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 \\
2-\sqrt6+\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2+2\sqrt2 & 4+2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 10 & 2+\sqrt6+\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2+2\sqrt2 & 2 \\
-2 & 2-\sqrt6+\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2+2\sqrt2 & 4+2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 10 & 2+\sqrt6+\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2+2\sqrt2 \\
2-2\sqrt2 & -2 & 2-\sqrt6+\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2+2\sqrt2 & 4+2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 10 & 2+\sqrt6+\sqrt2+2\sqrt3 & 4+2\sqrt3 \\
4-2\sqrt3 & 2-2\sqrt2 & -2 & 2-\sqrt6+\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2+2\sqrt2 & 4+2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 10 & 2+\sqrt6+\sqrt2+2\sqrt3 \\
2-\sqrt6-\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2-2\sqrt2 & -2 & 2-\sqrt6+\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2+2\sqrt2 & 4+2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 10
\end{bmatrix}
$$

$k = 3,$
$C = \frac{1}{48}$ of

((Cols. 1-12))

$$
\left[\begin{matrix}
14 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2 & 4-2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 \\
2+\sqrt6+3\sqrt2+2\sqrt3 & 14 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2 & 4-2\sqrt3 \\
4+2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 14 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2 \\
2 & 4+2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 14 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 \\
-2 & 2 & 4+2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 14 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 \\
2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2 & 4+2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 14 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 \\
-2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2 & 4+2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 14 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 \\
2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2 & 4+2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 14 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2 & -2 \\
2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2 & 4+2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 14 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2 \\
2 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2 & 4+2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 14 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 4+2\sqrt3 \\
4-2\sqrt3 & 2 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2 & 4+2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 14 & 2+\sqrt6+3\sqrt2+2\sqrt3 \\
2-\sqrt6-3\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2 & 4+2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 14 \\
-2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2 & 4+2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 \\
2-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-3\sqrt3+2\sqrt3 & 4-2\sqrt3 & 2 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2 & 4+2\sqrt3 \\
4-2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2 \\
2 & 4-2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 \\
2 & 2 & 4-2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 \\
2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2 & 4-2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 \\
-2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2 & 4-2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 \\
2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2 & 4-2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2 & 2 \\
-2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2 & 4-2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2 \\
2 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2 & 4-2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 4-2\sqrt3 \\
4+2\sqrt3 & 2 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2 & 4-2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-3\sqrt2+2\sqrt3 \\
2+\sqrt6+3\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2 & 4-2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -2
\end{matrix}\right.
$$

(Cols. 13 - 24)

$$
\begin{bmatrix}
-2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2 & 4+2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 \\
2-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2 & 4+2\sqrt3 \\
4-2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2 \\
2 & 4-2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 \\
2 & 2 & 4-2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 \\
2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2 & 4-2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 \\
-2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2 & 4-2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 \\
2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2 & 4-2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2 & 2 \\
-2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2 & 4-2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2 \\
2 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2 & 4-2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 4-2\sqrt3 \\
4+2\sqrt3 & 2 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2 & 4-2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-3\sqrt2+2\sqrt3 \\
2+\sqrt6+3\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2 & 4-2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -2 \\
14 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2 & 4-2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 \\
2+\sqrt6+3\sqrt2+2\sqrt3 & 14 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2 & 4-2\sqrt3 \\
4+2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 14 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2 \\
2 & 4+2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 14 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 \\
-2 & 2 & 4+2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 14 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 \\
2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2 & 4+2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 14 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 \\
-2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2 & 4+2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 14 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 \\
2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2 & 4+2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 14 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2 & -2 \\
2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2 & 4+2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 14 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 4+2\sqrt3 & 2 \\
2 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2 & 4+2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 14 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 4+2\sqrt3 \\
4-2\sqrt3 & 2 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2 & 4+2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 14 & 2+\sqrt6+3\sqrt2+2\sqrt3 \\
2-\sqrt6-3\sqrt2+2\sqrt3 & 4-2\sqrt3 & 2 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 2 & 4+2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 14
\end{bmatrix}
$$

$k = 4,$

$C = \frac{1}{48}$ of

(Cols. 1-12)

$$\left[\begin{array}{cccccccccccc}
18 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 2+2\sqrt3 & -2 & -4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 & 2-2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 \\
4+\sqrt6+3\sqrt2+2\sqrt3 & 18 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 2+2\sqrt3 & -2 & -4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 & 2-2\sqrt3 \\
2+2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 18 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 2+2\sqrt3 & -2 & -4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 \\
-2 & 2+2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 18 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 2+2\sqrt3 & -2 & -4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot \\
-4 & -2 & 2+2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 18 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 2+2\sqrt3 & -2 & -4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 4-\sqrt6+3\sqrt2-2\sqrt3 \\
4+\sqrt6-3\sqrt2-2\sqrt3 & -4 & -2 & 2+2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 18 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 2+2\sqrt3 & -2 & -4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 2 \\
2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -4 & -2 & 2+2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 18 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 2+2\sqrt3 & -2 & -4 & 4+\sqrt6-3\sqrt2-2\sqrt3 \\
4-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -4 & -2 & 2+2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 18 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 2+2\sqrt3 & -2 & -4 \\
\cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -4 & -2 & 2+2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 18 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 2+2\sqrt3 & -2 \\
-2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -4 & -2 & 2+2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 18 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 2+2\sqrt3 \\
2-2\sqrt3 & -2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -4 & -2 & 2+2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 18 & 4+\sqrt6+3\sqrt2+2\sqrt3 \\
4-\sqrt6-3\sqrt2+2\sqrt3 & 2-2\sqrt3 & -2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -4 & -2 & 2+2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 18 \\
2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & 2-2\sqrt3 & -2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -4 & -2 & 2+2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 \\
4-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & 2-2\sqrt3 & -2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -4 & -2 & 2+2\sqrt3 \\
2-2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & 2-2\sqrt3 & -2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -4 & -2 \\
-2 & 2-2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & 2-2\sqrt3 & -2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -4 \\
\cdot & -2 & 2-2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & 2-2\sqrt3 & -2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 4+\sqrt6-3\sqrt2-2\sqrt3 \\
4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 & 2-2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & 2-2\sqrt3 & -2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & 2 \\
2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 & 2-2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & 2-2\sqrt3 & -2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 \\
4+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 & 2-2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & 2-2\sqrt3 & -2 & \cdot \\
-4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 & 2-2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & 2-2\sqrt3 & -2 \\
-2 & -4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 & 2-2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & 2-2\sqrt3 \\
2+2\sqrt3 & -2 & -4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 & 2-2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 4-\sqrt6-3\sqrt2+2\sqrt3 \\
4+\sqrt6+3\sqrt2+2\sqrt3 & 2+2\sqrt3 & -2 & -4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 & 2-2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & 2
\end{array}\right.$$

(Cols. 13-24)

2	$4-\sqrt6-3\sqrt2+2\sqrt3$	$2-2\sqrt3$	-2	$\cdot$	$4-\sqrt6+3\sqrt2-2\sqrt3$	2	$4+\sqrt6+3\sqrt2-2\sqrt3$	-4	-2	$2+2\sqrt3$	$4+\sqrt6+3\sqrt2+2\sqrt3$
$4-\sqrt6-3\sqrt2+2\sqrt3$	2	$4-\sqrt6-3\sqrt2+2\sqrt3$	$2-2\sqrt3$	-2	$\cdot$	$4-\sqrt6+3\sqrt2-2\sqrt3$	2	$4+\sqrt6+3\sqrt2-2\sqrt3$	-4	-2	$2+2\sqrt3$
$2-2\sqrt3$	$4-\sqrt6-3\sqrt2+2\sqrt3$	2	$4-\sqrt6-3\sqrt2+2\sqrt3$	$2-2\sqrt3$	-2	$\cdot$	$4-\sqrt6+3\sqrt2-2\sqrt3$	2	$4+\sqrt6+3\sqrt2-2\sqrt3$	-4	-2
-2	$2-2\sqrt3$	$4-\sqrt6-3\sqrt2+2\sqrt3$	2	$4-\sqrt6-3\sqrt2+2\sqrt3$	$2-2\sqrt3$	-2	$\cdot$	$4-\sqrt6+3\sqrt2-2\sqrt3$	2	$4+\sqrt6+3\sqrt2-2\sqrt3$	-4
$\cdot$	-2	$2-2\sqrt3$	$4-\sqrt6-3\sqrt2+2\sqrt3$	2	$4-\sqrt6-3\sqrt2+2\sqrt3$	$2-2\sqrt3$	-2	$\cdot$	$4-\sqrt6+3\sqrt2-2\sqrt3$	2	$4+\sqrt6+3\sqrt2-2\sqrt3$
$4-\sqrt6+3\sqrt2-2\sqrt3$	$\cdot$	-2	$2-2\sqrt3$	$4-\sqrt6-3\sqrt2+2\sqrt3$	2	$4-\sqrt6-3\sqrt2+2\sqrt3$	$2-2\sqrt3$	-2	$\cdot$	$4-\sqrt6+3\sqrt2-2\sqrt3$	2
2	$4-\sqrt6+3\sqrt2-2\sqrt3$	$\cdot$	-2	$2-2\sqrt3$	$4-\sqrt6-3\sqrt2+2\sqrt3$	2	$4-\sqrt6-3\sqrt2+2\sqrt3$	$2-2\sqrt3$	-2	$\cdot$	$4-\sqrt6+3\sqrt2-2\sqrt3$
$4+\sqrt6+3\sqrt2-2\sqrt3$	2	$4-\sqrt6+3\sqrt2-2\sqrt3$	$\cdot$	-2	$2-2\sqrt3$	$4-\sqrt6-3\sqrt2+2\sqrt3$	2	$4-\sqrt6-3\sqrt2+2\sqrt3$	$2-2\sqrt3$	-2	$\cdot$
-4	$4+\sqrt6+3\sqrt2-2\sqrt3$	2	$4-\sqrt6+3\sqrt2-2\sqrt3$	$\cdot$	-2	$2-2\sqrt3$	$4-\sqrt6-3\sqrt2+2\sqrt3$	2	$4-\sqrt6-3\sqrt2+2\sqrt3$	$2-2\sqrt3$	-2
-2	-4	$4+\sqrt6+3\sqrt2-2\sqrt3$	2	$4-\sqrt6+3\sqrt2-2\sqrt3$	$\cdot$	-2	$2-2\sqrt3$	$4-\sqrt6-3\sqrt2+2\sqrt3$	2	$4-\sqrt6-3\sqrt2+2\sqrt3$	$2-2\sqrt3$
$2+2\sqrt3$	-2	-4	$4+\sqrt6+3\sqrt2-2\sqrt3$	2	$4-\sqrt6+3\sqrt2-2\sqrt3$	$\cdot$	-2	$2-2\sqrt3$	$4-\sqrt6-3\sqrt2+2\sqrt3$	2	$4-\sqrt6-3\sqrt2+2\sqrt3$
$4+\sqrt6+3\sqrt2+2\sqrt3$	$2+2\sqrt3$	-2	-4	$4+\sqrt6+3\sqrt2-2\sqrt3$	2	$4-\sqrt6+3\sqrt2-2\sqrt3$	$\cdot$	-2	$2-2\sqrt3$	$4-\sqrt6-3\sqrt2+2\sqrt3$	2
18	$4+\sqrt6+3\sqrt2+2\sqrt3$	$2+2\sqrt3$	-2	-4	$4+\sqrt6+3\sqrt2-2\sqrt3$	2	$4-\sqrt6+3\sqrt2-2\sqrt3$	$\cdot$	-2	$2-2\sqrt3$	$4-\sqrt6-3\sqrt2+2\sqrt3$
$4+\sqrt6+3\sqrt2+2\sqrt3$	18	$4+\sqrt6+3\sqrt2+2\sqrt3$	$2+2\sqrt3$	-2	-4	$4+\sqrt6+3\sqrt2-2\sqrt3$	2	$4-\sqrt6+3\sqrt2-2\sqrt3$	$\cdot$	-2	$2-2\sqrt3$
$2+2\sqrt3$	$4+\sqrt6+3\sqrt2+2\sqrt3$	18	$4+\sqrt6+3\sqrt2+2\sqrt3$	$2+2\sqrt3$	-2	-4	$4+\sqrt6+3\sqrt2-2\sqrt3$	2	$4-\sqrt6+3\sqrt2-2\sqrt3$	$\cdot$	-2
-2	$2+2\sqrt3$	$4+\sqrt6+3\sqrt2+2\sqrt3$	18	$4+\sqrt6+3\sqrt2+2\sqrt3$	$2+2\sqrt3$	-2	-4	$4+\sqrt6+3\sqrt2-2\sqrt3$	2	$4-\sqrt6+3\sqrt2-2\sqrt3$	$\cdot$
-4	-2	$2+2\sqrt3$	$4+\sqrt6+3\sqrt2+2\sqrt3$	18	$4+\sqrt6+3\sqrt2+2\sqrt3$	$2+2\sqrt3$	-2	-4	$4+\sqrt6+3\sqrt2-2\sqrt3$	2	$4-\sqrt6+3\sqrt2-2\sqrt3$
$4+\sqrt6+3\sqrt2-2\sqrt3$	-4	-2	$2+2\sqrt3$	$4+\sqrt6+3\sqrt2+2\sqrt3$	18	$4+\sqrt6+3\sqrt2+2\sqrt3$	$2+2\sqrt3$	-2	-4	$4+\sqrt6+3\sqrt2-2\sqrt3$	2
2	$4+\sqrt6+3\sqrt2-2\sqrt3$	-4	-2	$2+2\sqrt3$	$4+\sqrt6+3\sqrt2+2\sqrt3$	18	$4+\sqrt6+3\sqrt2+2\sqrt3$	$2+2\sqrt3$	-2	-4	$4+\sqrt6+3\sqrt2-2\sqrt3$
$4-\sqrt6+3\sqrt2-2\sqrt3$	2	$4+\sqrt6+3\sqrt2-2\sqrt3$	-4	-2	$2+2\sqrt3$	$4+\sqrt6+3\sqrt2+2\sqrt3$	18	$4+\sqrt6+3\sqrt2+2\sqrt3$	$2+2\sqrt3$	-2	-4
$\cdot$	$4-\sqrt6+3\sqrt2-2\sqrt3$	2	$4+\sqrt6+3\sqrt2-2\sqrt3$	-4	-2	$2+2\sqrt3$	$4+\sqrt6+3\sqrt2+2\sqrt3$	18	$4+\sqrt6+3\sqrt2+2\sqrt3$	$2+2\sqrt3$	-2
-2	$\cdot$	$4-\sqrt6+3\sqrt2-2\sqrt3$	2	$4+\sqrt6+3\sqrt2-2\sqrt3$	-4	-2	$2+2\sqrt3$	$4+\sqrt6+3\sqrt2+2\sqrt3$	18	$4+\sqrt6+3\sqrt2+2\sqrt3$	$2+2\sqrt3$
$2-2\sqrt3$	-2	$\cdot$	$4-\sqrt6+3\sqrt2-2\sqrt3$	2	$4+\sqrt6+3\sqrt2-2\sqrt3$	-4	-2	$2+2\sqrt3$	$4+\sqrt6+3\sqrt2+2\sqrt3$	18	$4+\sqrt6+3\sqrt2+2\sqrt3$
$4-\sqrt6-3\sqrt2+2\sqrt3$	$2-2\sqrt3$	-2	$\cdot$	$4-\sqrt6+3\sqrt2-2\sqrt3$	2	$4+\sqrt6+3\sqrt2-2\sqrt3$	-4	-2	$2+2\sqrt3$	$4+\sqrt6+3\sqrt2+2\sqrt3$	18

$k = 5,$

$C = \frac{1}{48}$ of

(Cols. 1- 12)

$$\begin{bmatrix}
22 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 2 & -2-2\sqrt{2} & -2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & 2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & -2 & -2+2\sqrt{2} & 2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} \\
4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 22 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 2 & -2-2\sqrt{2} & -2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & 2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & -2 & -2+2\sqrt{2} & 2 \\
2 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 22 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 2 & -2-2\sqrt{2} & -2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & 2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & -2 & -2+2\sqrt{2} \\
-2-2\sqrt{2} & 2 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 22 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 2 & -2-2\sqrt{2} & -2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & 2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & -2 \\
-2 & -2-2\sqrt{2} & 2 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 22 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 2 & -2-2\sqrt{2} & -2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & 2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} \\
4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & -2 & -2-2\sqrt{2} & 2 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 22 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 2 & -2-2\sqrt{2} & -2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & 2 \\
2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & -2 & -2-2\sqrt{2} & 2 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 22 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 2 & -2-2\sqrt{2} & -2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} \\
4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & 2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & -2 & -2-2\sqrt{2} & 2 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 22 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 2 & -2-2\sqrt{2} & -2 \\
-2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & 2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & -2 & -2-2\sqrt{2} & 2 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 22 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 2 & -2-2\sqrt{2} \\
-2+2\sqrt{2} & -2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & 2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & -2 & -2-2\sqrt{2} & 2 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 22 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 2 \\
2 & -2+2\sqrt{2} & -2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & 2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & -2 & -2-2\sqrt{2} & 2 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 22 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} \\
4-2\sqrt{6}-2\sqrt{3}+2\sqrt{3} & 2 & -2+2\sqrt{2} & -2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & 2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & -2 & -2-2\sqrt{2} & 2 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 22 \\
-2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & 2 & -2+2\sqrt{2} & -2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & 2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & -2 & -2-2\sqrt{2} & 2 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} \\
4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & -2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & 2 & -2+2\sqrt{2} & -2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & 2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & -2 & -2-2\sqrt{2} & 2 \\
2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & -2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & 2 & -2+2\sqrt{2} & -2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & 2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & -2 & -2-2\sqrt{2} \\
-2+2\sqrt{2} & 2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & -2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & 2 & -2+2\sqrt{2} & -2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & 2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & -2 \\
-2 & -2+2\sqrt{2} & 2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & -2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & 2 & -2+2\sqrt{2} & -2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & 2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} \\
4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & -2 & -2+2\sqrt{2} & 2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & -2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & 2 & -2+2\sqrt{2} & -2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & 2 \\
2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & -2 & -2+2\sqrt{2} & 2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & -2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & 2 & -2+2\sqrt{2} & -2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} \\
4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & 2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & -2 & -2+2\sqrt{2} & 2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & -2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & 2 & -2+2\sqrt{2} & -2 \\
-2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & 2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & -2 & -2+2\sqrt{2} & 2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & -2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & 2 & -2+2\sqrt{2} \\
-2-2\sqrt{2} & -2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & 2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & -2 & -2+2\sqrt{2} & 2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & -2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & 2 \\
2 & -2-2\sqrt{2} & -2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & 2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & -2 & -2+2\sqrt{2} & 2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & -2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} \\
4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 2 & -2-2\sqrt{2} & -2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & 2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & -2 & -2+2\sqrt{2} & 2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & -2
\end{bmatrix}$$

(Cols. 13- 24)

$$
\left.\begin{matrix}
-2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & 2 & -2+2\sqrt{2} & -2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & 2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & -2 & -2-2\sqrt{2} & 2 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} \\
4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & -2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & 2 & -2+2\sqrt{2} & -2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & 2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & -2 & -2-2\sqrt{2} & 2 \\
2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & -2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & 2 & -2+2\sqrt{2} & -2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & 2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & -2 & -2-2\sqrt{2} \\
-2+2\sqrt{2} & 2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & -2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & 2 & -2+2\sqrt{2} & -2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & 2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & -2 \\
-2 & -2+2\sqrt{2} & 2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & -2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & 2 & -2+2\sqrt{2} & -2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & 2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} \\
4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & -2 & -2+2\sqrt{2} & 2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & -2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & 2 & -2+2\sqrt{2} & -2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & 2 \\
2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & -2 & -2+2\sqrt{2} & 2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & -2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & 2 & -2+2\sqrt{2} & -2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} \\
4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & 2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & -2 & -2+2\sqrt{2} & 2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & -2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & 2 & -2+2\sqrt{2} & -2 \\
-2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & 2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & -2 & -2+2\sqrt{2} & 2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & -2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & 2 & -2+2\sqrt{2} \\
-2-2\sqrt{2} & -2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & 2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & -2 & -2+2\sqrt{2} & 2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & -2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & 2 \\
2 & -2-2\sqrt{2} & -2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & 2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & -2 & -2+2\sqrt{2} & 2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & -2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} \\
4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 2 & -2-2\sqrt{2} & -2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & 2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & -2 & -2+2\sqrt{2} & 2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & -2 \\
22 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 2 & -2-2\sqrt{2} & -2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & 2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & -2 & -2+2\sqrt{2} & 2 & 4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} \\
4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 22 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 2 & -2-2\sqrt{2} & -2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & 2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & -2 & -2+2\sqrt{2} & 2 \\
2 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 22 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 2 & -2-2\sqrt{2} & -2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & 2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & -2 & -2+2\sqrt{2} \\
-2-2\sqrt{2} & 2 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 22 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 2 & -2-2\sqrt{2} & -2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & 2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & -2 \\
-2 & -2-2\sqrt{2} & 2 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 22 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 2 & -2-2\sqrt{2} & -2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & 2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} \\
4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & -2 & -2-2\sqrt{2} & 2 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 22 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 2 & -2-2\sqrt{2} & -2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & 2 \\
2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & -2 & -2-2\sqrt{2} & 2 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 22 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 2 & -2-2\sqrt{2} & -2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} \\
4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & 2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & -2 & -2-2\sqrt{2} & 2 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 22 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 2 & -2-2\sqrt{2} & -2 \\
-2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & 2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & -2 & -2-2\sqrt{2} & 2 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 22 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 2 & -2-2\sqrt{2} \\
-2+2\sqrt{2} & -2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & 2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & -2 & -2-2\sqrt{2} & 2 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 22 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 2 \\
2 & -2+2\sqrt{2} & -2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & 2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & -2 & -2-2\sqrt{2} & 2 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 22 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} \\
4-2\sqrt{6}-2\sqrt{2}+2\sqrt{3} & 2 & -2+2\sqrt{2} & -2 & 4-2\sqrt{6}+2\sqrt{2}-2\sqrt{3} & 2 & 4+2\sqrt{6}-2\sqrt{2}-2\sqrt{3} & -2 & -2-2\sqrt{2} & 2 & 4+2\sqrt{6}+2\sqrt{2}+2\sqrt{3} & 22
\end{matrix}\right]
$$

$k = 6,$

$C = \frac{1}{48}$ of

(Cols 1-12)

$$\left[\begin{array}{cccccccccccc}
26 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & -2 & -2-2\sqrt2 & 2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & -2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & 2 & -2+2\sqrt2 & -2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 \\
4+2\sqrt6+2\sqrt2+2\sqrt3 & 26 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & -2 & -2-2\sqrt2 & 2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & -2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & 2 & -2+2\sqrt2 & -2 \\
-2 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & 26 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & -2 & -2-2\sqrt2 & 2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & -2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & 2 & -2+2\sqrt2 \\
-2-2\sqrt2 & -2 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & 26 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & -2 & -2-2\sqrt2 & 2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & -2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & 2 \\
2 & -2-2\sqrt2 & -2 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & 26 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & -2 & -2-2\sqrt2 & 2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & -2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 \\
4+2\sqrt6-2\sqrt2-2\sqrt3 & 2 & -2-2\sqrt2 & -2 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & 26 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & -2 & -2-2\sqrt2 & 2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & -2 \\
-2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & 2 & -2-2\sqrt2 & -2 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & 26 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & -2 & -2-2\sqrt2 & 2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 \\
4-2\sqrt6+2\sqrt2-2\sqrt3 & -2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & 2 & -2-2\sqrt2 & -2 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & 26 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & -2 & -2-2\sqrt2 & 2 \\
2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & -2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & 2 & -2-2\sqrt2 & -2 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & 26 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & -2 & -2-2\sqrt2 \\
-2+2\sqrt2 & 2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & -2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & 2 & -2-2\sqrt2 & -2 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & 26 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & -2 \\
-2 & -2+2\sqrt2 & 2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & -2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & 2 & -2-2\sqrt2 & -2 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & 26 & 4+2\sqrt6+2\sqrt2+2\sqrt3 \\
4-2\sqrt6-2\sqrt2+2\sqrt3 & -2 & -2+2\sqrt2 & 2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & -2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & 2 & -2-2\sqrt2 & -2 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & 26 \\
2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & -2 & -2+2\sqrt2 & 2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & -2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & 2 & -2-2\sqrt2 & -2 & 4+2\sqrt6+2\sqrt2+2\sqrt3 \\
4-2\sqrt6-2\sqrt2+2\sqrt3 & 2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & -2 & -2+2\sqrt2 & 2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & -2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & 2 & -2-2\sqrt2 & -2 \\
-2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & 2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & -2 & -2+2\sqrt2 & 2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & -2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & 2 & -2-2\sqrt2 \\
-2+2\sqrt2 & -2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & 2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & -2 & -2+2\sqrt2 & 2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & -2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & 2 \\
2 & -2+2\sqrt2 & -2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & 2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & -2 & -2+2\sqrt2 & 2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & -2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 \\
4-2\sqrt6+2\sqrt2-2\sqrt3 & 2 & -2+2\sqrt2 & -2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & 2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & -2 & -2+2\sqrt2 & 2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & -2 \\
-2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & 2 & -2+2\sqrt2 & -2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & 2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & -2 & -2+2\sqrt2 & 2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 \\
4+2\sqrt6-2\sqrt2-2\sqrt3 & -2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & 2 & -2+2\sqrt2 & -2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & 2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & -2 & -2+2\sqrt2 & 2 \\
2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & -2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & 2 & -2+2\sqrt2 & -2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & 2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & -2 & -2+2\sqrt2 \\
-2-2\sqrt2 & 2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & -2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & 2 & -2+2\sqrt2 & -2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & 2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & -2 \\
-2 & -2-2\sqrt2 & 2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & -2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & 2 & -2+2\sqrt2 & -2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & 2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 \\
4+2\sqrt6+2\sqrt2+2\sqrt3 & -2 & -2-2\sqrt2 & 2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & -2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & 2 & -2+2\sqrt2 & -2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & 2
\end{array}\right.$$

(Cols. 13 - 24)

$$
\left.\begin{matrix}
2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & -2 & -2+2\sqrt2 & 2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & -2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & 2 & -2-2\sqrt2 & -2 & 4+2\sqrt6+2\sqrt2+2\sqrt3 \\
4-2\sqrt6-2\sqrt2+2\sqrt3 & 2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & -2 & -2+2\sqrt2 & 2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & -2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & 2 & -2-2\sqrt2 & -2 \\
-2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & 2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & -2 & -2+2\sqrt2 & 2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & -2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & 2 & -2-2\sqrt2 \\
-2+2\sqrt2 & -2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & 2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & -2 & -2+2\sqrt2 & 2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & -2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & 2 \\
2 & -2+2\sqrt2 & -2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & 2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & -2 & -2+2\sqrt2 & 2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & -2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 \\
4-2\sqrt6+2\sqrt2-2\sqrt3 & 2 & -2+2\sqrt2 & -2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & 2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & -2 & -2+2\sqrt2 & 2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & -2 \\
-2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & 2 & -2+2\sqrt2 & -2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & 2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & -2 & -2+2\sqrt2 & 2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 \\
4+2\sqrt6-2\sqrt2-2\sqrt3 & -2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & 2 & -2+2\sqrt2 & -2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & 2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & -2 & -2+2\sqrt2 & 2 \\
2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & -2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & 2 & -2+2\sqrt2 & -2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & 2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & -2 & -2+2\sqrt2 \\
-2-2\sqrt2 & 2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & -2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & 2 & -2+2\sqrt2 & -2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & 2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & -2 \\
-2 & -2-2\sqrt2 & 2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & -2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & 2 & -2+2\sqrt2 & -2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & 2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 \\
4+2\sqrt6+2\sqrt2+2\sqrt3 & -2 & -2-2\sqrt2 & 2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & -2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & 2 & -2+2\sqrt2 & -2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 & 2 \\
26 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & -2 & -2-2\sqrt2 & 2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & -2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & 2 & -2+2\sqrt2 & -2 & 4-2\sqrt6-2\sqrt2+2\sqrt3 \\
4+2\sqrt6+2\sqrt2+2\sqrt3 & 26 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & -2 & -2-2\sqrt2 & 2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & -2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & 2 & -2+2\sqrt2 & -2 \\
-2 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & 26 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & -2 & -2-2\sqrt2 & 2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & -2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & 2 & -2+2\sqrt2 \\
-2-2\sqrt2 & -2 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & 26 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & -2 & -2-2\sqrt2 & 2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & -2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & 2 \\
2 & -2-2\sqrt2 & -2 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & 26 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & -2 & -2-2\sqrt2 & 2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & -2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 \\
4+2\sqrt6-2\sqrt2-2\sqrt3 & 2 & -2-2\sqrt2 & -2 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & 26 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & -2 & -2-2\sqrt2 & 2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & -2 \\
-2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & 2 & -2-2\sqrt2 & -2 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & 26 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & -2 & -2-2\sqrt2 & 2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 \\
4-2\sqrt6+2\sqrt2-2\sqrt3 & -2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & 2 & -2-2\sqrt2 & -2 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & 26 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & -2 & -2-2\sqrt2 & 2 \\
2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & -2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & 2 & -2-2\sqrt2 & -2 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & 26 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & -2 & -2-2\sqrt2 \\
-2+2\sqrt2 & 2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & -2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & 2 & -2-2\sqrt2 & -2 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & 26 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & -2 \\
-2 & -2+2\sqrt2 & 2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & -2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & 2 & -2-2\sqrt2 & -2 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & 26 & 4+2\sqrt6+2\sqrt2+2\sqrt3 \\
4-2\sqrt6-2\sqrt2+2\sqrt3 & -2 & -2+2\sqrt2 & 2 & 4-2\sqrt6+2\sqrt2-2\sqrt3 & -2 & 4+2\sqrt6-2\sqrt2-2\sqrt3 & 2 & -2-2\sqrt2 & -2 & 4+2\sqrt6+2\sqrt2+2\sqrt3 & 26
\end{matrix}\right]
$$

$k = 7,$
$C = \frac{1}{48}$ of

(Cols 1-12)

$$
\left[\begin{matrix}
30 & 4+\sqrt6+3\sqrt2+2\sqrt3 & -2-2\sqrt3 & -2 & 4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 & -2+2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 \\
4+\sqrt6+3\sqrt2+2\sqrt3 & 30 & 4+\sqrt6+3\sqrt2+2\sqrt3 & -2-2\sqrt3 & -2 & 4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 & -2+2\sqrt3 \\
-2-2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 30 & 4+\sqrt6+3\sqrt2+2\sqrt3 & -2-2\sqrt3 & -2 & 4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 \\
-2 & -2-2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 30 & 4+\sqrt6+3\sqrt2+2\sqrt3 & -2-2\sqrt3 & -2 & 4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot \\
4 & -2 & -2-2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 30 & 4+\sqrt6+3\sqrt2+2\sqrt3 & -2-2\sqrt3 & -2 & 4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 4-\sqrt6+3\sqrt2-2\sqrt3 \\
4+\sqrt6-3\sqrt2-2\sqrt3 & 4 & -2 & -2-2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 30 & 4+\sqrt6+3\sqrt2+2\sqrt3 & -2-2\sqrt3 & -2 & 4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -2 \\
-2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 4 & -2 & -2-2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 30 & 4+\sqrt6+3\sqrt2+2\sqrt3 & -2-2\sqrt3 & -2 & 4 & 4+\sqrt6-3\sqrt2-2\sqrt3 \\
4-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 4 & -2 & -2-2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 30 & 4+\sqrt6+3\sqrt2+2\sqrt3 & -2-2\sqrt3 & -2 & 4 \\
\cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 4 & -2 & -2-2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 30 & 4+\sqrt6+3\sqrt2+2\sqrt3 & -2-2\sqrt3 & -2 \\
-2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 4 & -2 & -2-2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 30 & 4+\sqrt6+3\sqrt2+2\sqrt3 & -2-2\sqrt3 \\
-2+2\sqrt3 & -2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 4 & -2 & -2-2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 30 & 4+\sqrt6+3\sqrt2+2\sqrt3 \\
4-\sqrt6-3\sqrt2+2\sqrt3 & -2+2\sqrt3 & -2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 4 & -2 & -2-2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 30 \\
-2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2+2\sqrt3 & -2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 4 & -2 & -2-2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 \\
4-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2+2\sqrt3 & -2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 4 & -2 & -2-2\sqrt3 \\
-2+2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2+2\sqrt3 & -2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 4 & -2 \\
-2 & -2+2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2+2\sqrt3 & -2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 4 \\
\cdot & -2 & -2+2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2+2\sqrt3 & -2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 4+\sqrt6-3\sqrt2-2\sqrt3 \\
4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 & -2+2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2+2\sqrt3 & -2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & -2 \\
-2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 & -2+2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2+2\sqrt3 & -2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 \\
4+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 & -2+2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2+2\sqrt3 & -2 & \cdot \\
4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 & -2+2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2+2\sqrt3 & -2 \\
-2 & 4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 & -2+2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2+2\sqrt3 \\
-2-2\sqrt3 & -2 & 4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 & -2+2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 4-\sqrt6-3\sqrt2+2\sqrt3 \\
4+\sqrt6+3\sqrt2+2\sqrt3 & -2-2\sqrt3 & -2 & 4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 & -2+2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2
\end{matrix}\right.
$$

(Cols. 13-24)

$$
\left.\begin{matrix}
-2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2+2\sqrt3 & -2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 4 & -2 & -2-2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 \\
4-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2+2\sqrt3 & -2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 4 & -2 & -2-2\sqrt3 \\
-2+2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2+2\sqrt3 & -2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 4 & -2 \\
-2 & -2+2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2+2\sqrt3 & -2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 4 \\
\cdot & -2 & -2+2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2+2\sqrt3 & -2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 4+\sqrt6-3\sqrt2-2\sqrt3 \\
4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 & -2+2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2+2\sqrt3 & -2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & -2 \\
-2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 & -2+2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2+2\sqrt3 & -2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 \\
4+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 & -2+2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2+2\sqrt3 & -2 & \cdot \\
4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 & -2+2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2+2\sqrt3 & -2 \\
-2 & 4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 & -2+2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2+2\sqrt3 \\
-2-2\sqrt3 & -2 & 4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 & -2+2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2 & 4-\sqrt6-3\sqrt2+2\sqrt3 \\
4+\sqrt6+3\sqrt2+2\sqrt3 & -2-2\sqrt3 & -2 & 4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 & -2+2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 & -2 \\
30 & 4+\sqrt6+3\sqrt2+2\sqrt3 & -2-2\sqrt3 & -2 & 4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 & -2+2\sqrt3 & 4-\sqrt6-3\sqrt2+2\sqrt3 \\
4+\sqrt6+3\sqrt2+2\sqrt3 & 30 & 4+\sqrt6+3\sqrt2+2\sqrt3 & -2-2\sqrt3 & -2 & 4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 & -2+2\sqrt3 \\
-2-2\sqrt3 & 4+\sqrt6+3\sqrt3+2\sqrt3 & 30 & 4+\sqrt6+3\sqrt2+2\sqrt3 & -2-2\sqrt3 & -2 & 4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot & -2 \\
-2 & -2-2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 30 & 4+\sqrt6+3\sqrt2+2\sqrt3 & -2-2\sqrt3 & -2 & 4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 4-\sqrt6+3\sqrt2-2\sqrt3 & \cdot \\
4 & -2 & -2-2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 30 & 4+\sqrt6+3\sqrt2+2\sqrt3 & -2-2\sqrt3 & -2 & 4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -2 & 4-\sqrt6+3\sqrt2-2\sqrt3 \\
4+\sqrt6-3\sqrt2-2\sqrt3 & 4 & -2 & -2-2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 30 & 4+\sqrt6+3\sqrt2+2\sqrt3 & -2-2\sqrt3 & -2 & 4 & 4+\sqrt6-3\sqrt2-2\sqrt3 & -2 \\
-2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 4 & -2 & -2-2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 30 & 4+\sqrt6+3\sqrt2+2\sqrt3 & -2-2\sqrt3 & -2 & 4 & 4+\sqrt6-3\sqrt2-2\sqrt3 \\
4-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 4 & -2 & -2-2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 30 & 4+\sqrt6+3\sqrt2+2\sqrt3 & -2-2\sqrt3 & -2 & 4 \\
\cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 4 & -2 & -2-2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 30 & 4+\sqrt6+3\sqrt2+2\sqrt3 & -2-2\sqrt3 & -2 \\
-2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 4 & -2 & -2-2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 30 & 4+\sqrt6+3\sqrt2+2\sqrt3 & -2-2\sqrt3 \\
-2+2\sqrt3 & -2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 4 & -2 & -2-2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 30 & 4+\sqrt6+3\sqrt2+2\sqrt3 \\
4-\sqrt6-3\sqrt2+2\sqrt3 & -2+2\sqrt3 & -2 & \cdot & 4-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 4+\sqrt6-3\sqrt2-2\sqrt3 & 4 & -2 & -2-2\sqrt3 & 4+\sqrt6+3\sqrt2+2\sqrt3 & 30
\end{matrix}\right]
$$

$k = 8,$

$C = \frac{1}{48}$ of

(Cols. 1–12)

$$
\left[\begin{array}{cccccccccccc}
34 & 2+\sqrt6+3\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2 & -4+2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 \\
2+\sqrt6+3\sqrt2+2\sqrt3 & 34 & 2+\sqrt6+3\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2 & -4+2\sqrt3 \\
-4-2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 34 & 2+\sqrt6+3\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2 \\
2 & -4-2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 34 & 2+\sqrt6+3\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 \\
2 & 2 & -4-2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 34 & 2+\sqrt6+3\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 \\
2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2 & -4-2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 34 & 2+\sqrt6+3\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 \\
2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2 & -4-2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 34 & 2+\sqrt6+3\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 \\
2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2 & -4-2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 34 & 2+\sqrt6+3\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2 & 2 \\
-2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2 & -4-2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 34 & 2+\sqrt6+3\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2 \\
2 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2 & -4-2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 34 & 2+\sqrt6+3\sqrt2+2\sqrt3 & -4-2\sqrt3 \\
-4+2\sqrt3 & 2 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2 & -4-2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 34 & 2+\sqrt6+3\sqrt2+2\sqrt3 \\
2-\sqrt6-3\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2 & -4-2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 34 \\
2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2 & -4-2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 \\
2-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2 & -4-2\sqrt3 \\
-4+2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2 \\
2 & -4+2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 \\
-2 & 2 & -4+2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 \\
2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2 & -4+2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 \\
2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2 & -4+2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 \\
2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2 & -4+2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2 & -2 \\
2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2 & -4+2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2 \\
2 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2 & -4+2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -4+2\sqrt3 \\
-4-2\sqrt3 & 2 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2 & -4+2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-3\sqrt2+2\sqrt3 \\
2+\sqrt6+3\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2 & -4+2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 2
\end{array}\right.
$$

(Cols. 13 – 24)

$$
\left.\begin{matrix}
2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2 & -4-2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 \\
2-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2 & -4-2\sqrt3 \\
-4+2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2 \\
2 & -4+2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 \\
-2 & 2 & -4+2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 \\
2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2 & -4+2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 \\
2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2 & -4+2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 \\
2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2 & -4+2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2 & -2 \\
2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2 & -4+2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2 \\
2 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2 & -4+2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-3\sqrt2+2\sqrt3 & -4+2\sqrt3 \\
-4-2\sqrt3 & 2 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2 & -4+2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 2 & 2-\sqrt6-3\sqrt2+2\sqrt3 \\
2+\sqrt6+3\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2 & -4+2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 & 2 \\
34 & 2+\sqrt6+3\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2 & -4+2\sqrt3 & 2-\sqrt6-3\sqrt2+2\sqrt3 \\
2+\sqrt6+3\sqrt2+2\sqrt3 & 34 & 2+\sqrt6+3\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2 & -4+2\sqrt3 \\
-4-2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 34 & 2+\sqrt6+3\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 & 2 \\
2 & -4-2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 34 & 2+\sqrt6+3\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & -2 \\
2 & 2 & -4-2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 34 & 2+\sqrt6+3\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+3\sqrt2-2\sqrt3 \\
2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2 & -4-2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 34 & 2+\sqrt6+3\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 \\
2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2 & -4-2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 34 & 2+\sqrt6+3\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 \\
2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2 & -4-2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 34 & 2+\sqrt6+3\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2 & 2 \\
-2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2 & -4-2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 34 & 2+\sqrt6+3\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2 \\
2 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2 & -4-2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 34 & 2+\sqrt6+3\sqrt2+2\sqrt3 & -4-2\sqrt3 \\
-4+2\sqrt3 & 2 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2 & -4-2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 34 & 2+\sqrt6+3\sqrt2+2\sqrt3 \\
2-\sqrt6-3\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2 & -2 & 2-\sqrt6+3\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-3\sqrt2-2\sqrt3 & 2 & 2 & -4-2\sqrt3 & 2+\sqrt6+3\sqrt2+2\sqrt3 & 34
\end{matrix}\right]
$$

$k = 9,$

$C = \frac{1}{48}$ of

(Cols. 1-12)

$$
\begin{bmatrix}
38 & 2+\sqrt6+\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2+2\sqrt2 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2-2\sqrt2 & -4+2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 \\
2+\sqrt6+\sqrt2+2\sqrt3 & 38 & 2+\sqrt6+\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2+2\sqrt2 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2-2\sqrt2 & -4+2\sqrt3 \\
-4-2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 38 & 2+\sqrt6+\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2+2\sqrt2 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2-2\sqrt2 \\
2+2\sqrt2 & -4-2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 38 & 2+\sqrt6+\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2+2\sqrt2 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 \\
-2 & 2+2\sqrt2 & -4-2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 38 & 2+\sqrt6+\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2+2\sqrt2 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 \\
2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2+2\sqrt2 & -4-2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 38 & 2+\sqrt6+\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2+2\sqrt2 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 \\
2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2+2\sqrt2 & -4-2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 38 & 2+\sqrt6+\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2+2\sqrt2 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 \\
2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2+2\sqrt2 & -4-2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 38 & 2+\sqrt6+\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2+2\sqrt2 & -2 \\
2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2+2\sqrt2 & -4-2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 38 & 2+\sqrt6+\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2+2\sqrt2 \\
2-2\sqrt2 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2+2\sqrt2 & -4-2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 38 & 2+\sqrt6+\sqrt2+2\sqrt3 & -4-2\sqrt3 \\
-4+2\sqrt3 & 2-2\sqrt2 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2+2\sqrt2 & -4-2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 38 & 2+\sqrt6+\sqrt2+2\sqrt3 \\
2-\sqrt6-\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2-2\sqrt2 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2+2\sqrt2 & -4-2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 38 \\
-2 & 2-\sqrt6-\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2-2\sqrt2 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2+2\sqrt2 & -4-2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 \\
2-\sqrt6-\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2-2\sqrt2 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2+2\sqrt2 & -4-2\sqrt3 \\
-4+2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2-2\sqrt2 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2+2\sqrt2 \\
2-2\sqrt2 & -4+2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2-2\sqrt2 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 \\
2 & 2-2\sqrt2 & -4+2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2-2\sqrt2 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 \\
2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2-2\sqrt2 & -4+2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2-2\sqrt2 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 \\
2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2-2\sqrt2 & -4+2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2-2\sqrt2 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 \\
2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2-2\sqrt2 & -4+2\sqrt3 & 2-\sqrt6-\sqrt3+2\sqrt3 & -2 & 2-\sqrt6-\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2-2\sqrt2 & 2 \\
-2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2-2\sqrt2 & -4+2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2-2\sqrt2 \\
2+2\sqrt2 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2-2\sqrt2 & -4+2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-\sqrt2+2\sqrt3 & -4+2\sqrt3 \\
-4-2\sqrt3 & 2+2\sqrt2 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2-2\sqrt2 & -4+2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-\sqrt2+2\sqrt3 \\
2+\sqrt6+\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2+2\sqrt2 & -2 & 2+\sqrt6-\sqrt3-2\sqrt3 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2-2\sqrt2 & -4+2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & -2
\end{bmatrix}
$$

(Cols. 13 - 24)

$$\left.\begin{matrix}
-2 & 2-\sqrt6-\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2-2\sqrt2 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2+2\sqrt2 & -4-2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 \\
2-\sqrt6-\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2-2\sqrt2 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2+2\sqrt2 & -4-2\sqrt3 \\
-4+2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2-2\sqrt2 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2+2\sqrt2 \\
2-2\sqrt2 & -4+2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2-2\sqrt2 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 \\
2 & 2-2\sqrt2 & -4+2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2-2\sqrt2 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 \\
2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2-2\sqrt2 & -4+2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2-2\sqrt2 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 \\
2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2-2\sqrt2 & -4+2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2-2\sqrt2 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 \\
2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2-2\sqrt2 & -4+2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2-2\sqrt2 & 2 \\
-2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2-2\sqrt2 & -4+2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2-2\sqrt2 \\
2+2\sqrt2 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2-2\sqrt2 & -4+2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-\sqrt2+2\sqrt3 & -4+2\sqrt3 \\
-4-2\sqrt3 & 2+2\sqrt2 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2-2\sqrt2 & -4+2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & -2 & 2-\sqrt6-\sqrt2+2\sqrt3 \\
2+\sqrt6+\sqrt3+2\sqrt3 & -4-2\sqrt3 & 2+2\sqrt2 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2-2\sqrt2 & -4+2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 & -2 \\
38 & 2+\sqrt6+\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2+2\sqrt2 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2-2\sqrt2 & -4+2\sqrt3 & 2-\sqrt6-\sqrt2+2\sqrt3 \\
2+\sqrt6+\sqrt2+2\sqrt3 & 38 & 2+\sqrt6+\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2+2\sqrt2 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2-2\sqrt2 & -4+2\sqrt3 \\
-4-2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 38 & 2+\sqrt6+\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2+2\sqrt2 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2-2\sqrt2 \\
2+2\sqrt2 & -4-2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 38 & 2+\sqrt6+\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2+2\sqrt2 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 \\
-2 & 2+2\sqrt2 & -4-2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 38 & 2+\sqrt6+\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2+2\sqrt2 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 \\
2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2+2\sqrt2 & -4-2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 38 & 2+\sqrt6+\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2+2\sqrt2 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 & 2 \\
2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2+2\sqrt2 & -4-2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 38 & 2+\sqrt6+\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2+2\sqrt2 & -2 & 2+\sqrt6-\sqrt2-2\sqrt3 \\
2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2+2\sqrt2 & -4-2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 38 & 2+\sqrt6+\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2+2\sqrt2 & -2 \\
2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2+2\sqrt2 & -4-2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 38 & 2+\sqrt6+\sqrt2+2\sqrt3 & -4-2\sqrt3 & 2+2\sqrt2 \\
2-2\sqrt2 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2+2\sqrt2 & -4-2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 38 & 2+\sqrt6+\sqrt2+2\sqrt3 & -4-2\sqrt3 \\
-4+2\sqrt3 & 2-2\sqrt2 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2+2\sqrt2 & -4-2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 38 & 2+\sqrt6+\sqrt2+2\sqrt3 \\
2-\sqrt6-\sqrt2+2\sqrt3 & -4+2\sqrt3 & 2-2\sqrt2 & 2 & 2-\sqrt6+\sqrt2-2\sqrt3 & 2 & 2+\sqrt6-\sqrt2-2\sqrt3 & -2 & 2+2\sqrt2 & -4-2\sqrt3 & 2+\sqrt6+\sqrt2+2\sqrt3 & 38
\end{matrix}\right]$$

$k = 10,$
$C = \frac{1}{48}$ of

(Cols. 1- 12)

$$
\left[\begin{array}{cccccccccccc}
42 & 2+\sqrt{6}+\sqrt{2} & -2-2\sqrt{3} & 2+2\sqrt{2} & -4 & 2+\sqrt{6}-\sqrt{2} & -2 & 2-\sqrt{6}+\sqrt{2} & \cdot & 2-2\sqrt{2} & -2+2\sqrt{3} & 2-\sqrt{6}-\sqrt{2} \\
2+\sqrt{6}+\sqrt{2} & 42 & 2+\sqrt{6}+\sqrt{2} & -2-2\sqrt{3} & 2+2\sqrt{2} & -4 & 2+\sqrt{6}-\sqrt{2} & -2 & 2-\sqrt{6}+\sqrt{2} & \cdot & 2-2\sqrt{2} & -2+2\sqrt{3} \\
-2-2\sqrt{3} & 2+\sqrt{6}+\sqrt{2} & 42 & 2+\sqrt{6}+\sqrt{2} & -2-2\sqrt{3} & 2+2\sqrt{2} & -4 & 2+\sqrt{6}-\sqrt{2} & -2 & 2-\sqrt{6}+\sqrt{2} & \cdot & 2-2\sqrt{2} \\
2+2\sqrt{2} & -2-2\sqrt{3} & 2+\sqrt{6}+\sqrt{2} & 42 & 2+\sqrt{6}+\sqrt{2} & -2-2\sqrt{3} & 2+2\sqrt{2} & -4 & 2+\sqrt{6}-\sqrt{2} & -2 & 2-\sqrt{6}+\sqrt{2} & \cdot \\
-4 & 2+2\sqrt{2} & -2-2\sqrt{3} & 2+\sqrt{6}+\sqrt{2} & 42 & 2+\sqrt{6}+\sqrt{2} & -2-2\sqrt{3} & 2+2\sqrt{2} & -4 & 2+\sqrt{6}-\sqrt{2} & -2 & 2-\sqrt{6}+\sqrt{2} \\
2+\sqrt{6}-\sqrt{2} & -4 & 2+2\sqrt{2} & -2-2\sqrt{3} & 2+\sqrt{6}+\sqrt{2} & 42 & 2+\sqrt{6}+\sqrt{2} & -2-2\sqrt{3} & 2+2\sqrt{2} & -4 & 2+\sqrt{6}-\sqrt{2} & -2 \\
-2 & 2+\sqrt{6}-\sqrt{2} & -4 & 2+2\sqrt{2} & -2-2\sqrt{3} & 2+\sqrt{6}+\sqrt{2} & 42 & 2+\sqrt{6}+\sqrt{2} & -2-2\sqrt{3} & 2+2\sqrt{2} & -4 & 2+\sqrt{6}-\sqrt{2} \\
2-\sqrt{6}+\sqrt{2} & -2 & 2+\sqrt{6}-\sqrt{2} & -4 & 2+2\sqrt{2} & -2-2\sqrt{3} & 2+\sqrt{6}+\sqrt{2} & 42 & 2+\sqrt{6}+\sqrt{2} & -2-2\sqrt{3} & 2+2\sqrt{2} & -4 \\
\cdot & 2-\sqrt{6}+\sqrt{2} & -2 & 2+\sqrt{6}-\sqrt{2} & -4 & 2+2\sqrt{2} & -2-2\sqrt{3} & 2+\sqrt{6}+\sqrt{2} & 42 & 2+\sqrt{6}+\sqrt{2} & -2-2\sqrt{3} & 2+2\sqrt{2} \\
2-2\sqrt{2} & \cdot & 2-\sqrt{6}+\sqrt{2} & -2 & 2+\sqrt{6}-\sqrt{2} & -4 & 2+2\sqrt{2} & -2-2\sqrt{3} & 2+\sqrt{6}+\sqrt{2} & 42 & 2+\sqrt{6}+\sqrt{2} & -2-2\sqrt{3} \\
-2+2\sqrt{3} & 2-2\sqrt{2} & \cdot & 2-\sqrt{6}+\sqrt{2} & -2 & 2+\sqrt{6}-\sqrt{2} & -4 & 2+2\sqrt{2} & -2-2\sqrt{3} & 2+\sqrt{6}+\sqrt{2} & 42 & 2+\sqrt{6}+\sqrt{2} \\
2-\sqrt{6}-\sqrt{2} & -2+2\sqrt{3} & 2-2\sqrt{2} & \cdot & 2-\sqrt{6}+\sqrt{2} & -2 & 2+\sqrt{6}-\sqrt{2} & -4 & 2+2\sqrt{2} & -2-2\sqrt{3} & 2+\sqrt{6}+\sqrt{2} & 42 \\
2 & 2-\sqrt{6}-\sqrt{2} & -2+2\sqrt{3} & 2-2\sqrt{2} & \cdot & 2-\sqrt{6}+\sqrt{2} & -2 & 2+\sqrt{6}-\sqrt{2} & -4 & 2+2\sqrt{2} & -2-2\sqrt{3} & 2+\sqrt{6}+\sqrt{2} \\
2-\sqrt{6}-\sqrt{2} & 2 & 2-\sqrt{6}-\sqrt{2} & -2+2\sqrt{3} & 2-2\sqrt{2} & \cdot & 2-\sqrt{6}+\sqrt{2} & -2 & 2+\sqrt{6}-\sqrt{2} & -4 & 2+2\sqrt{2} & -2-2\sqrt{3} \\
-2+2\sqrt{3} & 2-\sqrt{6}-\sqrt{2} & 2 & 2-\sqrt{6}-\sqrt{2} & -2+2\sqrt{3} & 2-2\sqrt{2} & \cdot & 2-\sqrt{6}+\sqrt{2} & -2 & 2+\sqrt{6}-\sqrt{2} & -4 & 2+2\sqrt{2} \\
2-2\sqrt{2} & -2+2\sqrt{3} & 2-\sqrt{6}-\sqrt{2} & 2 & 2-\sqrt{6}-\sqrt{2} & -2+2\sqrt{3} & 2-2\sqrt{2} & \cdot & 2-\sqrt{6}+\sqrt{2} & -2 & 2+\sqrt{6}-\sqrt{2} & -4 \\
\cdot & 2-2\sqrt{2} & -2+2\sqrt{3} & 2-\sqrt{6}-\sqrt{2} & 2 & 2-\sqrt{6}-\sqrt{2} & -2+2\sqrt{3} & 2-2\sqrt{2} & \cdot & 2-\sqrt{6}+\sqrt{2} & -2 & 2+\sqrt{6}-\sqrt{2} \\
2-\sqrt{6}+\sqrt{2} & \cdot & 2-2\sqrt{2} & -2+2\sqrt{3} & 2-\sqrt{6}-\sqrt{2} & 2 & 2-\sqrt{6}-\sqrt{2} & -2+2\sqrt{3} & 2-2\sqrt{2} & \cdot & 2-\sqrt{6}+\sqrt{2} & -2 \\
-2 & 2-\sqrt{6}+\sqrt{2} & \cdot & 2-2\sqrt{2} & -2+2\sqrt{3} & 2-\sqrt{6}-\sqrt{2} & 2 & 2-\sqrt{6}-\sqrt{2} & -2+2\sqrt{3} & 2-2\sqrt{2} & \cdot & 2-\sqrt{6}+\sqrt{2} \\
2+\sqrt{6}-\sqrt{2} & -2 & 2-\sqrt{6}+\sqrt{2} & \cdot & 2-2\sqrt{2} & -2+2\sqrt{3} & 2-\sqrt{6}-\sqrt{2} & 2 & 2-\sqrt{6}-\sqrt{2} & -2+2\sqrt{3} & 2-2\sqrt{2} & \cdot \\
-4 & 2+\sqrt{6}-\sqrt{2} & -2 & 2-\sqrt{6}+\sqrt{2} & \cdot & 2-2\sqrt{2} & -2+2\sqrt{3} & 2-\sqrt{6}-\sqrt{2} & 2 & 2-\sqrt{6}-\sqrt{2} & -2+2\sqrt{3} & 2-2\sqrt{2} \\
2+2\sqrt{2} & -4 & 2+\sqrt{6}-\sqrt{2} & -2 & 2-\sqrt{6}+\sqrt{2} & \cdot & 2-2\sqrt{2} & -2+2\sqrt{3} & 2-\sqrt{6}-\sqrt{2} & 2 & 2-\sqrt{6}-\sqrt{2} & -2+2\sqrt{3} \\
-2-2\sqrt{3} & 2+2\sqrt{2} & -4 & 2+\sqrt{6}-\sqrt{2} & -2 & 2-\sqrt{6}+\sqrt{2} & \cdot & 2-2\sqrt{2} & -2+2\sqrt{3} & 2-\sqrt{6}-\sqrt{2} & 2 & 2-\sqrt{6}-\sqrt{2} \\
2+\sqrt{6}+\sqrt{2} & -2-2\sqrt{3} & 2+2\sqrt{2} & -4 & 2+\sqrt{6}-\sqrt{2} & -2 & 2-\sqrt{6}+\sqrt{2} & \cdot & 2-2\sqrt{2} & -2+2\sqrt{3} & 2-\sqrt{6}-\sqrt{2} & 2
\end{array}\right.
$$

(Cols. 13 – 24)

$$
\begin{bmatrix}
2 & 2-\sqrt6-\sqrt2 & -2+2\sqrt3 & 2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 & -2 & 2+\sqrt6-\sqrt2 & -4 & 2+2\sqrt2 & -2-2\sqrt3 & 2+\sqrt6+\sqrt2 \\
2-\sqrt6-\sqrt2 & 2 & 2-\sqrt6-\sqrt2 & -2+2\sqrt3 & 2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 & -2 & 2+\sqrt6-\sqrt2 & -4 & 2+2\sqrt2 & -2-2\sqrt3 \\
-2+2\sqrt3 & 2-\sqrt6-\sqrt2 & 2 & 2-\sqrt6-\sqrt2 & -2+2\sqrt3 & 2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 & -2 & 2+\sqrt6-\sqrt2 & -4 & 2+2\sqrt2 \\
2-2\sqrt2 & -2+2\sqrt3 & 2-\sqrt6-\sqrt2 & 2 & 2-\sqrt6-\sqrt2 & -2+2\sqrt3 & 2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 & -2 & 2+\sqrt6-\sqrt2 & -4 \\
\cdot & 2-2\sqrt2 & -2+2\sqrt3 & 2-\sqrt6-\sqrt2 & 2 & 2-\sqrt6-\sqrt2 & -2+2\sqrt3 & 2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 & -2 & 2+\sqrt6-\sqrt2 \\
2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 & -2+2\sqrt3 & 2-\sqrt6-\sqrt2 & 2 & 2-\sqrt6-\sqrt2 & -2+2\sqrt3 & 2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 & -2 \\
-2 & 2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 & -2+2\sqrt3 & 2-\sqrt6-\sqrt2 & 2 & 2-\sqrt6-\sqrt2 & -2+2\sqrt3 & 2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 \\
2+\sqrt6-\sqrt2 & -2 & 2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 & -2+2\sqrt3 & 2-\sqrt6-\sqrt2 & 2 & 2-\sqrt6-\sqrt2 & -2+2\sqrt3 & 2-2\sqrt2 & \cdot \\
-4 & 2+\sqrt6-\sqrt2 & -2 & 2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 & -2+2\sqrt3 & 2-\sqrt6-\sqrt2 & 2 & 2-\sqrt6-\sqrt2 & -2+2\sqrt3 & 2-2\sqrt2 \\
2+2\sqrt2 & -4 & 2+\sqrt6-\sqrt2 & -2 & 2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 & -2+2\sqrt3 & 2-\sqrt6-\sqrt2 & 2 & 2-\sqrt6-\sqrt2 & -2+2\sqrt3 \\
-2-2\sqrt3 & 2+2\sqrt2 & -4 & 2+\sqrt6-\sqrt2 & -2 & 2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 & -2+2\sqrt3 & 2-\sqrt6-\sqrt2 & 2 & 2-\sqrt6-\sqrt2 \\
2+\sqrt6+\sqrt2 & -2-2\sqrt3 & 2+2\sqrt2 & -4 & 2+\sqrt6-\sqrt2 & -2 & 2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 & -2+2\sqrt3 & 2-\sqrt6-\sqrt2 & 2 \\
42 & 2+\sqrt6+\sqrt2 & -2-2\sqrt3 & 2+2\sqrt2 & -4 & 2+\sqrt6-\sqrt2 & -2 & 2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 & -2+2\sqrt3 & 2-\sqrt6-\sqrt2 \\
2+\sqrt6+\sqrt2 & 42 & 2+\sqrt6+\sqrt2 & -2-2\sqrt3 & 2+2\sqrt2 & -4 & 2+\sqrt6-\sqrt2 & -2 & 2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 & -2+2\sqrt3 \\
-2-2\sqrt3 & 2+\sqrt6+\sqrt2 & 42 & 2+\sqrt6+\sqrt2 & -2-2\sqrt3 & 2+2\sqrt2 & -4 & 2+\sqrt6-\sqrt2 & -2 & 2-\sqrt6+\sqrt2 & \cdot & 2-2\sqrt2 \\
2+2\sqrt2 & -2-2\sqrt3 & 2+\sqrt6+\sqrt2 & 42 & 2+\sqrt6+\sqrt2 & -2-2\sqrt3 & 2+2\sqrt2 & -4 & 2+\sqrt6-\sqrt2 & -2 & 2-\sqrt6+\sqrt2 & \cdot \\
-4 & 2+2\sqrt2 & -2-2\sqrt3 & 2+\sqrt6+\sqrt2 & 42 & 2+\sqrt6+\sqrt2 & -2-2\sqrt3 & 2+2\sqrt2 & -4 & 2+\sqrt6-\sqrt2 & -2 & 2-\sqrt6+\sqrt2 \\
2+\sqrt6-\sqrt2 & -4 & 2+2\sqrt2 & -2-2\sqrt3 & 2+\sqrt6+\sqrt2 & 42 & 2+\sqrt6+\sqrt2 & -2-2\sqrt3 & 2+2\sqrt2 & -4 & 2+\sqrt6-\sqrt2 & -2 \\
-2 & 2+\sqrt6-\sqrt2 & -4 & 2+2\sqrt2 & -2-2\sqrt3 & 2+\sqrt6+\sqrt2 & 42 & 2+\sqrt6+\sqrt2 & -2-2\sqrt3 & 2+2\sqrt2 & -4 & 2+\sqrt6-\sqrt2 \\
2-\sqrt6+\sqrt2 & -2 & 2+\sqrt6-\sqrt2 & -4 & 2+2\sqrt2 & -2-2\sqrt3 & 2+\sqrt6+\sqrt2 & 42 & 2+\sqrt6+\sqrt2 & -2-2\sqrt3 & 2+2\sqrt2 & -4 \\
\cdot & 2-\sqrt6+\sqrt2 & -2 & 2+\sqrt6-\sqrt2 & -4 & 2+2\sqrt2 & -2-2\sqrt3 & 2+\sqrt6+\sqrt2 & 42 & 2+\sqrt6+\sqrt2 & -2-2\sqrt3 & 2+2\sqrt2 \\
2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 & -2 & 2+\sqrt6-\sqrt2 & -4 & 2+2\sqrt2 & -2-2\sqrt3 & 2+\sqrt6+\sqrt2 & 42 & 2+\sqrt6+\sqrt2 & -2-2\sqrt3 \\
-2+2\sqrt3 & 2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 & -2 & 2+\sqrt6-\sqrt2 & -4 & 2+2\sqrt2 & -2-2\sqrt3 & 2+\sqrt6+\sqrt2 & 42 & 2+\sqrt6+\sqrt2 \\
2-\sqrt6-\sqrt2 & -2+2\sqrt3 & 2-2\sqrt2 & \cdot & 2-\sqrt6+\sqrt2 & -2 & 2+\sqrt6-\sqrt2 & -4 & 2+2\sqrt2 & -2-2\sqrt3 & 2+\sqrt6+\sqrt2 & 42
\end{bmatrix}
$$

$k = 11,$
$C = \frac{1}{24}$ of

$$\left[\begin{array}{*{24}{c}}
23 & 1 & -1 \\
1 & 23 & 1 & -1 \\
-1 & 1 & 23 & 1 & -1 \\
& -1 & 1 & 23 & 1 & -1 \\
&& -1 & 1 & 23 & 1 & -1 \\
&&& -1 & 1 & 23 & 1 & -1 \\
&&&& -1 & 1 & 23 & 1 & -1 \\
&&&&& -1 & 1 & 23 & 1 & -1 \\
&&&&&& -1 & 1 & 23 & 1 & -1 \\
&&&&&&& -1 & 1 & 23 & 1 & -1 \\
&&&&&&&& -1 & 1 & 23 & 1 & -1 \\
&&&&&&&&& -1 & 1 & 23 & 1 & -1 \\
&&&&&&&&&& -1 & 1 & 23 & 1 & -1 \\
&&&&&&&&&&& -1 & 1 & 23 & 1 & -1 \\
&&&&&&&&&&&& -1 & 1 & 23 & 1 & -1 \\
&&&&&&&&&&&&& -1 & 1 & 23 & 1 & -1 \\
&&&&&&&&&&&&&& -1 & 1 & 23 & 1 & -1 \\
&&&&&&&&&&&&&&& -1 & 1 & 23 & 1 & -1 \\
&&&&&&&&&&&&&&&& -1 & 1 & 23 & 1 & -1 \\
&&&&&&&&&&&&&&&&& -1 & 1 & 23 & 1 & -1 \\
&&&&&&&&&&&&&&&&&& -1 & 1 & 23 & 1 & -1 \\
&&&&&&&&&&&&&&&&&&& -1 & 1 & 23 & 1 & -1 \\
&&&&&&&&&&&&&&&&&&&& -1 & 1 & 23 & 1 \\
&&&&&&&&&&&&&&&&&&&&& -1 & 1 & 23
\end{array}\right]$$

etc., +1 and −1 alternately.

etc., +1 and −1 alternately.

$k = 12, \quad C = I.$

APPENDIX II

Matrices which premultiply the observed data to give the Fourier coefficients.

$2n = 4$

$$\frac{1}{4}\begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 2 & \cdot & -2 & \cdot \\ \cdot & 2 & \cdot & -2 \\ \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

$2n = 6$

$$\frac{1}{6}\begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 2 & 1 & -1 & -2 & -1 & 1 \\ \cdot & \sqrt{3} & \sqrt{3} & \cdot & -\sqrt{3} & -\sqrt{3} \\ 2 & -1 & -1 & 2 & -1 & -1 \\ \cdot & \sqrt{3} & -\sqrt{3} & \cdot & \sqrt{3} & -\sqrt{3} \\ \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

$2n = 8$

$$\frac{1}{8}\begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 2 & \sqrt{2} & \cdot & -\sqrt{2} & -2 & -\sqrt{2} & \cdot & \sqrt{2} \\ \cdot & \sqrt{2} & 2 & \sqrt{2} & \cdot & -\sqrt{2} & -2 & -\sqrt{2} \\ 2 & \cdot & -2 & \cdot & 2 & \cdot & -2 & \cdot \\ \cdot & 2 & \cdot & -2 & \cdot & 2 & \cdot & -2 \\ 2 & -\sqrt{2} & \cdot & \sqrt{2} & -2 & \sqrt{2} & \cdot & -\sqrt{2} \\ \cdot & \sqrt{2} & -2 & \sqrt{2} & \cdot & -\sqrt{2} & 2 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

$$\underline{2n = 10}$$

$$\frac{1}{10}\begin{bmatrix}
\sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 \\
2 & \frac{\sqrt5+1}{2} & \frac{\sqrt5-1}{2} & -\frac{(\sqrt5-1)}{2} & -\frac{(\sqrt5+1)}{2} & -2 & -\frac{(\sqrt5+1)}{2} & -\frac{(\sqrt5-1)}{2} & \frac{\sqrt5-1}{2} & \frac{\sqrt5+1}{2} \\
\cdot & \frac{\sqrt{10-2\sqrt5}}{2} & \frac{\sqrt{10+2\sqrt5}}{2} & \frac{\sqrt{10+2\sqrt5}}{2} & \frac{\sqrt{10-2\sqrt5}}{2} & \cdot & -\frac{\sqrt{10-2\sqrt5}}{2} & -\frac{\sqrt{10+2\sqrt5}}{2} & -\frac{\sqrt{10+2\sqrt5}}{2} & -\frac{\sqrt{10-2\sqrt5}}{2} \\
2 & \frac{\sqrt5-1}{2} & -\frac{(\sqrt5+1)}{2} & -\frac{(\sqrt5+1)}{2} & \frac{\sqrt5-1}{2} & 2 & \frac{\sqrt5-1}{2} & -\frac{(\sqrt5+1)}{2} & -\frac{(\sqrt5+1)}{2} & \frac{\sqrt5-1}{2} \\
\cdot & \frac{\sqrt{10+2\sqrt5}}{2} & \frac{\sqrt{10-2\sqrt5}}{2} & -\frac{\sqrt{10-2\sqrt5}}{2} & -\frac{\sqrt{10+2\sqrt5}}{2} & \cdot & \frac{\sqrt{10+2\sqrt5}}{2} & \frac{\sqrt{10-2\sqrt5}}{2} & -\frac{\sqrt{10-2\sqrt5}}{2} & -\frac{\sqrt{10+2\sqrt5}}{2} \\
2 & -\frac{(\sqrt5-1)}{2} & -\frac{(\sqrt5+1)}{2} & \frac{\sqrt5+1}{2} & \frac{\sqrt5-1}{2} & -2 & \frac{\sqrt5-1}{2} & \frac{\sqrt5+1}{2} & -\frac{(\sqrt5+1)}{2} & -\frac{(\sqrt5-1)}{2} \\
\cdot & \frac{\sqrt{10+2\sqrt5}}{2} & -\frac{\sqrt{10-2\sqrt5}}{2} & -\frac{\sqrt{10-2\sqrt5}}{2} & \frac{\sqrt{10+2\sqrt5}}{2} & \cdot & -\frac{\sqrt{10+2\sqrt5}}{2} & \frac{\sqrt{10-2\sqrt5}}{2} & \frac{\sqrt{10-2\sqrt5}}{2} & -\frac{\sqrt{10+2\sqrt5}}{2} \\
2 & -\frac{(\sqrt5+1)}{2} & \frac{\sqrt5-1}{2} & \frac{\sqrt5-1}{2} & -\frac{(\sqrt5+1)}{2} & 2 & -\frac{(\sqrt5+1)}{2} & \frac{\sqrt5-1}{2} & \frac{\sqrt5-1}{2} & -\frac{(\sqrt5+1)}{2} \\
\cdot & \frac{\sqrt{10-2\sqrt5}}{2} & -\frac{\sqrt{10+2\sqrt5}}{2} & \frac{\sqrt{10+2\sqrt5}}{2} & -\frac{\sqrt{10-2\sqrt5}}{2} & \cdot & \frac{\sqrt{10-2\sqrt5}}{2} & -\frac{\sqrt{10+2\sqrt5}}{2} & \frac{\sqrt{10+2\sqrt5}}{2} & -\frac{\sqrt{10-2\sqrt5}}{2} \\
\sqrt2 & -\sqrt2 & \sqrt2 & -\sqrt2 & \sqrt2 & -\sqrt2 & \sqrt2 & -\sqrt2 & \sqrt2 & -\sqrt2
\end{bmatrix}$$

$$\underline{2n = 12}$$

$$\frac{1}{12}\begin{bmatrix}
\sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 \\
2 & \sqrt3 & 1 & \cdot & -1 & -\sqrt3 & -2 & -\sqrt3 & -1 & \cdot & 1 & \sqrt3 \\
\cdot & 1 & \sqrt3 & 2 & \sqrt3 & 1 & \cdot & -1 & -\sqrt3 & -2 & -\sqrt3 & -1 \\
2 & 1 & -1 & -2 & -1 & 1 & 2 & 1 & -1 & -2 & -1 & 1 \\
\cdot & \sqrt3 & \sqrt3 & \cdot & -\sqrt3 & -\sqrt3 & \cdot & \sqrt3 & \sqrt3 & \cdot & -\sqrt3 & -\sqrt3 \\
2 & \cdot & -2 & \cdot & 2 & \cdot & -2 & \cdot & 2 & \cdot & -2 & \cdot \\
\cdot & 2 & \cdot & -2 & \cdot & 2 & \cdot & -2 & \cdot & 2 & \cdot & -2 \\
2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 & -1 \\
\cdot & \sqrt3 & -\sqrt3 & \cdot & \sqrt3 & -\sqrt3 & \cdot & \sqrt3 & -\sqrt3 & \cdot & \sqrt3 & -\sqrt3 \\
2 & -\sqrt3 & 1 & \cdot & -1 & \sqrt3 & -2 & \sqrt3 & -1 & \cdot & 1 & -\sqrt3 \\
\cdot & 1 & -\sqrt3 & 2 & -\sqrt3 & 1 & \cdot & -1 & \sqrt3 & -2 & \sqrt3 & -1 \\
\sqrt2 & -\sqrt2 & \sqrt2 & -\sqrt2 & \sqrt2 & -\sqrt2 & \sqrt2 & -\sqrt2 & \sqrt2 & -\sqrt2 & \sqrt2 & -\sqrt2
\end{bmatrix}$$

$\underline{2n = 16}$

$$\frac{1}{16}\left[\begin{array}{cccccccccccccccc}
\sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 \\
2 & \sqrt{2+\sqrt2} & \sqrt2 & \sqrt{2-\sqrt2} & \cdot & -\sqrt{2-\sqrt2} & -\sqrt2 & -\sqrt{2+\sqrt2} & -2 & -\sqrt{2+\sqrt2} & -\sqrt2 & -\sqrt{2-\sqrt2} & \cdot & \sqrt{2-\sqrt2} & \sqrt2 & \sqrt{2+\sqrt2} \\
\cdot & \sqrt{2-\sqrt2} & \sqrt2 & \sqrt{2+\sqrt2} & 2 & \sqrt{2+\sqrt2} & \sqrt2 & \sqrt{2-\sqrt2} & \cdot & -\sqrt{2-\sqrt2} & -\sqrt2 & -\sqrt{2+\sqrt2} & -2 & -\sqrt{2+\sqrt2} & -\sqrt2 & -\sqrt{2-\sqrt2} \\
2 & \sqrt2 & \cdot & -\sqrt2 & -2 & -\sqrt2 & \cdot & \sqrt2 & 2 & \sqrt2 & \cdot & -\sqrt2 & -2 & -\sqrt2 & \cdot & \sqrt2 \\
\cdot & \sqrt2 & 2 & \sqrt2 & \cdot & -\sqrt2 & -2 & -\sqrt2 & \cdot & \sqrt2 & 2 & \sqrt2 & \cdot & -\sqrt2 & -2 & -\sqrt2 \\
2 & \sqrt{2-\sqrt2} & -\sqrt2 & -\sqrt{2+\sqrt2} & \cdot & \sqrt{2+\sqrt2} & \sqrt2 & -\sqrt{2-\sqrt2} & -2 & -\sqrt{2-\sqrt2} & \sqrt2 & \sqrt{2+\sqrt2} & \cdot & -\sqrt{2+\sqrt2} & -\sqrt2 & \sqrt{2-\sqrt2} \\
\cdot & \sqrt{2+\sqrt2} & \sqrt2 & -\sqrt{2-\sqrt2} & -2 & -\sqrt{2-\sqrt2} & \sqrt2 & \sqrt{2+\sqrt2} & \cdot & -\sqrt{2+\sqrt2} & -\sqrt2 & \sqrt{2-\sqrt2} & 2 & \sqrt{2-\sqrt2} & -\sqrt2 & -\sqrt{2+\sqrt2} \\
2 & \cdot & -2 & \cdot & 2 & \cdot & -2 & \cdot & 2 & \cdot & -2 & \cdot & 2 & \cdot & -2 & \cdot \\
\cdot & 2 & \cdot & -2 & \cdot & 2 & \cdot & -2 & \cdot & 2 & \cdot & -2 & \cdot & 2 & \cdot & -2 \\
2 & -\sqrt{2-\sqrt2} & -\sqrt2 & \sqrt{2+\sqrt2} & \cdot & -\sqrt{2+\sqrt2} & \sqrt2 & \sqrt{2-\sqrt2} & -2 & \sqrt{2-\sqrt2} & \sqrt2 & -\sqrt{2+\sqrt2} & \cdot & \sqrt{2+\sqrt2} & -\sqrt2 & -\sqrt{2-\sqrt2} \\
\cdot & \sqrt{2+\sqrt2} & -\sqrt2 & -\sqrt{2-\sqrt2} & 2 & -\sqrt{2-\sqrt2} & -\sqrt2 & \sqrt{2+\sqrt2} & \cdot & -\sqrt{2+\sqrt2} & \sqrt2 & \sqrt{2-\sqrt2} & -2 & \sqrt{2-\sqrt2} & \sqrt2 & -\sqrt{2+\sqrt2} \\
2 & -\sqrt2 & \cdot & \sqrt2 & -2 & \sqrt2 & \cdot & -\sqrt2 & 2 & -\sqrt2 & \cdot & \sqrt2 & -2 & \sqrt2 & \cdot & -\sqrt2 \\
\cdot & \sqrt2 & -2 & \sqrt2 & \cdot & -\sqrt2 & 2 & -\sqrt2 & \cdot & \sqrt2 & -2 & \sqrt2 & \cdot & -\sqrt2 & 2 & -\sqrt2 \\
2 & -\sqrt{2+\sqrt2} & \sqrt2 & -\sqrt{2-\sqrt2} & \cdot & \sqrt{2-\sqrt2} & -\sqrt2 & \sqrt{2+\sqrt2} & -2 & \sqrt{2+\sqrt2} & -\sqrt2 & \sqrt{2-\sqrt2} & \cdot & -\sqrt{2-\sqrt2} & \sqrt2 & -\sqrt{2+\sqrt2} \\
\cdot & \sqrt{2-\sqrt2} & -\sqrt2 & \sqrt{2+\sqrt2} & -2 & \sqrt{2+\sqrt2} & -\sqrt2 & \sqrt{2-\sqrt2} & \cdot & -\sqrt{2-\sqrt2} & \sqrt2 & -\sqrt{2+\sqrt2} & 2 & -\sqrt{2+\sqrt2} & \sqrt2 & -\sqrt{2-\sqrt2} \\
\sqrt2 & -\sqrt2 & \sqrt2 & -\sqrt2 & \sqrt2 & -\sqrt2 & \sqrt2 & -\sqrt2 & \sqrt2 & -\sqrt2 & \sqrt2 & -\sqrt2 & \sqrt2 & -\sqrt2 & \sqrt2 & -\sqrt2
\end{array}\right]$$

$2n = 24$

$\frac{1}{24}$ of

(Cols. 1–12)

$$
\begin{bmatrix}
\sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 \\
2 & \frac{\sqrt6+\sqrt2}{2} & \sqrt3 & \sqrt2 & 1 & \frac{\sqrt6-\sqrt2}{2} & \cdot & -\frac{(\sqrt6-\sqrt2)}{2} & -1 & -\sqrt2 & -\sqrt3 & -\frac{(\sqrt6+\sqrt2)}{2} \\
\cdot & \frac{\sqrt6-\sqrt2}{2} & 1 & \sqrt2 & \sqrt3 & \frac{\sqrt6+\sqrt2}{2} & 2 & \frac{\sqrt6+\sqrt2}{2} & \sqrt3 & \sqrt2 & 1 & \frac{\sqrt6-\sqrt2}{2} \\
2 & \sqrt3 & 1 & \cdot & -1 & -\sqrt3 & -2 & -\sqrt3 & -1 & \cdot & 1 & \sqrt3 \\
\cdot & 1 & \sqrt3 & 2 & \sqrt3 & 1 & \cdot & -1 & -\sqrt3 & -2 & -\sqrt3 & -1 \\
2 & \sqrt2 & \cdot & -\sqrt2 & -2 & -\sqrt2 & \cdot & \sqrt2 & 2 & \sqrt2 & \cdot & -\sqrt2 \\
\cdot & \sqrt2 & 2 & \sqrt2 & \cdot & -\sqrt2 & -2 & -\sqrt2 & \cdot & \sqrt2 & 2 & \sqrt2 \\
2 & 1 & -1 & -2 & -1 & 1 & 2 & 1 & -1 & -2 & -1 & 1 \\
\cdot & \sqrt3 & \sqrt3 & \cdot & -\sqrt3 & -\sqrt3 & \cdot & \sqrt3 & \sqrt3 & \cdot & -\sqrt3 & -\sqrt3 \\
2 & \frac{\sqrt6-\sqrt2}{2} & -\sqrt3 & -\sqrt2 & 1 & \frac{\sqrt6+\sqrt2}{2} & \cdot & -\frac{(\sqrt6+\sqrt2)}{2} & -1 & \sqrt2 & \sqrt3 & -\frac{(\sqrt6-\sqrt2)}{2} \\
\cdot & \frac{\sqrt6+\sqrt2}{2} & 1 & -\sqrt2 & -\sqrt3 & \frac{\sqrt6-\sqrt2}{2} & 2 & \frac{\sqrt6-\sqrt2}{2} & -\sqrt3 & -\sqrt2 & 1 & \frac{\sqrt6+\sqrt2}{2} \\
2 & \cdot & -2 & \cdot & 2 & \cdot & -2 & \cdot & 2 & \cdot & -2 & \cdot \\
\cdot & 2 & \cdot & -2 & \cdot & 2 & \cdot & -2 & \cdot & 2 & \cdot & -2 \\
2 & -\frac{(\sqrt6-\sqrt2)}{2} & -\sqrt3 & \sqrt2 & 1 & -\frac{(\sqrt6+\sqrt2)}{2} & \cdot & \frac{\sqrt6+\sqrt2}{2} & -1 & -\sqrt2 & \sqrt3 & \frac{\sqrt6-\sqrt2}{2} \\
\cdot & \frac{\sqrt6+\sqrt2}{2} & -1 & -\sqrt2 & \sqrt3 & \frac{\sqrt6-\sqrt2}{2} & -2 & \frac{\sqrt6-\sqrt2}{2} & \sqrt3 & -\sqrt2 & -1 & \frac{\sqrt6+\sqrt2}{2} \\
2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 & -1 \\
\cdot & \sqrt3 & -\sqrt3 & \cdot & \sqrt3 & -\sqrt3 & \cdot & \sqrt3 & -\sqrt3 & \cdot & \sqrt3 & -\sqrt3 \\
2 & -\sqrt2 & \cdot & \sqrt2 & -2 & \sqrt2 & \cdot & -\sqrt2 & 2 & -\sqrt2 & \cdot & \sqrt2 \\
\cdot & \sqrt2 & -2 & \sqrt2 & \cdot & -\sqrt2 & 2 & -\sqrt2 & \cdot & \sqrt2 & -2 & \sqrt2 \\
2 & -\sqrt3 & 1 & \cdot & -1 & \sqrt3 & -2 & \sqrt3 & -1 & \cdot & 1 & -\sqrt3 \\
\cdot & 1 & -\sqrt3 & 2 & -\sqrt3 & 1 & \cdot & -1 & \sqrt3 & -2 & \sqrt3 & -1 \\
2 & -\frac{(\sqrt6+\sqrt2)}{2} & \sqrt3 & -\sqrt2 & 1 & -\frac{(\sqrt6-\sqrt2)}{2} & \cdot & \frac{\sqrt6-\sqrt2}{2} & -1 & \sqrt2 & -\sqrt3 & \frac{\sqrt6+\sqrt2}{2} \\
\cdot & \frac{\sqrt6-\sqrt2}{2} & -1 & \sqrt2 & -\sqrt3 & \frac{\sqrt6+\sqrt2}{2} & -2 & \frac{\sqrt6+\sqrt2}{2} & -\sqrt3 & \sqrt2 & -1 & \frac{\sqrt6-\sqrt2}{2} \\
\sqrt2 & -\sqrt2 & \sqrt2 & -\sqrt2 & \sqrt2 & -\sqrt2 & \sqrt2 & -\sqrt2 & \sqrt2 & -\sqrt2 & \sqrt2 & -\sqrt2
\end{bmatrix}
$$

(Cols. 13 - 24).

$$
\left.\begin{array}{cccccccccccc}
\sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\
-2 & -\frac{(\sqrt{6}+\sqrt{2})}{2} & -\sqrt{3} & -\sqrt{2} & -1 & -\frac{(\sqrt{6}-\sqrt{2})}{2} & \cdot & \frac{\sqrt{6}-\sqrt{2}}{2} & 1 & \sqrt{2} & \sqrt{3} & \frac{\sqrt{6}+\sqrt{2}}{2} \\
\cdot & -\frac{(\sqrt{6}-\sqrt{2})}{2} & -1 & -\sqrt{2} & -\sqrt{3} & -\frac{(\sqrt{6}+\sqrt{2})}{2} & -2 & -\frac{(\sqrt{6}+\sqrt{2})}{2} & -\sqrt{3} & -\sqrt{2} & -1 & -\frac{(\sqrt{6}-\sqrt{2})}{2} \\
2 & \sqrt{3} & 1 & \cdot & -1 & -\sqrt{3} & -2 & -\sqrt{3} & -1 & \cdot & 1 & \sqrt{3} \\
\cdot & 1 & \sqrt{3} & \sqrt{2} & \sqrt{3} & 1 & \cdot & -1 & -\sqrt{3} & -\sqrt{2} & -\sqrt{3} & -1 \\
-2 & -\sqrt{2} & \cdot & \sqrt{2} & 2 & \sqrt{2} & \cdot & -\sqrt{2} & -2 & -\sqrt{2} & \cdot & \sqrt{2} \\
\cdot & -\sqrt{2} & -2 & -\sqrt{2} & \cdot & \sqrt{2} & 2 & \sqrt{2} & \cdot & -\sqrt{2} & -2 & -\sqrt{2} \\
2 & 1 & -1 & -2 & -1 & 1 & 2 & 1 & -1 & -2 & -1 & 1 \\
\cdot & \sqrt{3} & \sqrt{3} & \cdot & -\sqrt{3} & -\sqrt{3} & \cdot & \sqrt{3} & \sqrt{3} & \cdot & -\sqrt{3} & -\sqrt{3} \\
-2 & -\frac{(\sqrt{6}-\sqrt{2})}{2} & \sqrt{3} & \sqrt{2} & -1 & -\frac{(\sqrt{6}+\sqrt{2})}{2} & \cdot & \frac{\sqrt{6}+\sqrt{2}}{2} & 1 & -\sqrt{2} & -\sqrt{3} & \frac{\sqrt{6}-\sqrt{2}}{2} \\
\cdot & -\frac{(\sqrt{6}+\sqrt{2})}{2} & -1 & \sqrt{2} & \sqrt{3} & -\frac{(\sqrt{6}-\sqrt{2})}{2} & -2 & -\frac{(\sqrt{6}-\sqrt{2})}{2} & \sqrt{3} & \sqrt{2} & -1 & -\frac{(\sqrt{6}+\sqrt{2})}{2} \\
2 & \cdot & -2 & \cdot & 2 & \cdot & -2 & \cdot & 2 & \cdot & -2 & \cdot \\
\cdot & 2 & \cdot & -2 & \cdot & 2 & \cdot & -2 & \cdot & 2 & \cdot & -2 \\
-2 & \frac{\sqrt{6}-\sqrt{2}}{2} & \sqrt{3} & -\sqrt{2} & -1 & \frac{\sqrt{6}+\sqrt{2}}{2} & \cdot & -\frac{(\sqrt{6}+\sqrt{2})}{2} & 1 & \sqrt{2} & -\sqrt{3} & -\frac{(\sqrt{6}-\sqrt{2})}{2} \\
\cdot & -\frac{(\sqrt{6}+\sqrt{2})}{2} & 1 & \sqrt{2} & -\sqrt{3} & -\frac{(\sqrt{6}-\sqrt{2})}{2} & 2 & -\frac{(\sqrt{6}-\sqrt{2})}{2} & -\sqrt{3} & \sqrt{2} & 1 & -\frac{(\sqrt{6}+\sqrt{2})}{2} \\
2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 & -1 \\
\cdot & \sqrt{3} & -\sqrt{3} & \cdot & \sqrt{3} & -\sqrt{3} & \cdot & \sqrt{3} & -\sqrt{3} & \cdot & \sqrt{3} & -\sqrt{3} \\
-2 & \sqrt{2} & \cdot & -\sqrt{2} & 2 & -\sqrt{2} & \cdot & -\sqrt{2} & -2 & \sqrt{2} & \cdot & -\sqrt{2} \\
\cdot & -\sqrt{2} & 2 & -\sqrt{2} & \cdot & \sqrt{2} & -2 & \sqrt{2} & \cdot & -\sqrt{2} & 2 & -\sqrt{2} \\
2 & -\sqrt{3} & 1 & \cdot & -1 & \sqrt{3} & -2 & \sqrt{3} & -1 & \cdot & 1 & -\sqrt{3} \\
\cdot & 1 & -\sqrt{3} & 2 & -\sqrt{3} & 1 & \cdot & -1 & \sqrt{3} & -2 & \sqrt{3} & -1 \\
-2 & \frac{\sqrt{6}+\sqrt{2}}{2} & -\sqrt{3} & \sqrt{2} & -1 & \frac{\sqrt{6}-\sqrt{2}}{2} & \cdot & -\frac{(\sqrt{6}-\sqrt{2})}{2} & 1 & -\sqrt{2} & \sqrt{3} & -\frac{(\sqrt{6}+\sqrt{2})}{2} \\
\cdot & -\frac{(\sqrt{6}-\sqrt{2})}{2} & 1 & -\sqrt{2} & \sqrt{3} & -\frac{(\sqrt{6}+\sqrt{2})}{2} & 2 & -\frac{(\sqrt{6}+\sqrt{2})}{2} & \sqrt{3} & -\sqrt{2} & 1 & -\frac{(\sqrt{6}-\sqrt{2})}{2} \\
\sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2}
\end{array}\right]
$$

APPENDIX III.

ELEMENTS IN FIRST ROW OR COLUMN OF C

(for those matrices whose elements previously occurred in surd form).

2n = 8

k = 1	k = 2
3	5
2.414 214	2.414 214
1	-1
-0.414 214	-0.414 214
-1	1
-0.414 214	-0.414 214
1	-1
2.414 214	2.414 214

Matrix divided by 8.

2n = 10

k = 1	k = 2	k = 3
3	5	7
2.618 034	3.236 068	2.618 034
1.618 034	0	-1.618 034
0.381 966	-1.236 068	0.381 966
-0.618 034	0	0.618 034
-1	1	-1
-0.618 034	0	0.618 034
0.381 966	-1.236 068	0.381 966
1.618 034	0	-1.618 034
2.618 034	3.236 068	2.618 034

Matrix divided by 10.

2n = 12

k = 1	k = 2	k = 3	k = 4
3	5	7	9
2.732 051	3.732 051	3.732 051	2.732 051
2	1	-1	-2
1	-1	-1	1
0	1	1	0
-0.732 051	0.267 949	0.267 949	-0.732 051
-1	1	-1	1
-0.732 051	0.267 949	0.267 949	0.732 051
0	-1	1	0
1	-1	-1	1
2	1	-1	-2
2.732 051	3.732 051	3.732 051	2.732 051

Matrix divided by 12.

2n = 16

k = 1	k = 2	k = 3
3	5	7
2.847 759	4.261 432	5.026 799
2.414 214	2.414 214	1
1.765 367	0.351 153	-1.496 065
1	-1	-1
0.234 633	-1.179 581	0.667 637
-0.414 214	-0.414 214	1
-0.847 218	0.566 996	-0.198 371
-1	-3	-5
-0.847 218	0.566 996	-0.198 371
-0.414 214	-0.414 214	1
0.234 633	-1.179 581	0.667 637
1	-1	-1
1.765 367	0.351 153	-1.496 065
2.414 214	2.414 214	1
2.847 759	4.261 432	5.026 799

k = 4	k = 5	k = 6
9	11	13
5.026 799	4.261 432	2.847 759
-1	-2.414 214	-2.414 214
-1.496 065	0.351 153	1.765 367
1	1	-1
0.667 637	-1.179 581	0.234 633
-1	0.414 214	0.414 214
-0.198 371	0.566 996	-0.847 218
-3	-5	-3
-0.198 371	0.566 996	-0.847 218
-1	0.414 214	0.414 214
0.667 637	-1.179 581	0.234 633
1	1	-1
-1.496 065	0.351 153	1.765 367
-1	-2.414 214	-2.414 214
5.026 799	4.261 432	2.847 759

Matrix divided by 16.

2n = 24 /

$2n = 24$

$k = 1$	$k = 2$	$k = 3$	$k = 4$
3	5	7	9
2.931 852	4.663 903	6.078 117	7.078 117
2.732 051	3.732 051	3.732 051	2.732 051
2.414 214	2.414 214	1	-1
2	1	-1	-2
1.517 638	-0.214 413	-1.628 627	-0.628 627
1	-1	-1	1
0.482 362	-1.249 689	0.164 525	1.164 525
0	-1	1	0
-0.414 214	-0.414 214	1	-1
-0.732 051	0.267 949	0.267 949	-0.732 051
-0.931 852	0.800 199	-0.614 015	0.385 985
-1	1	-1	1
-0.931 852	0.800 199	-0.614 015	0.385 985
-0.732 051	0.267 949	0.267 949	-0.732 051
-0.414 214	-0.414 214	1	-1
0	-1	1	0
0.482 362	-1.249 689	0.164 525	1.164 525
1	-1	-1	1
1.517 638	-0.214 413	-1.628 627	-0.628 627
2	1	-1	-2
2.414 214	2.414 214	1	-1
2.732 051	3.732 051	3.732 051	2.732 051
2.931 852	4.663 903	6.078 117	7.078 117

$k = 5$	$k = 6$	$k = 7$	$k = 8$
11	13	15	17
7.595 755	7.595 755	7.078 117	6.078 117
1	-1	-2.732 051	-3.732 051
-2.414 214	-2.414 214	-1	1
-1	1	2	1
1.303 225	1.303 225	-0.628 627	-1.628 627
1	-1	-1	1
-0.767 327	-0.767 327	1.164 525	0.164 525
-1	1	0	-1
0.414 214	0.414 214	-1	1
1	-1	0.732 051	-0.267 949
-0.131 653	-0.131 653	0.385 985	-0.614 015
-1	1	-1	1
-0.131 653	-0.131 653	0.385 985	-0.614 015
1	-1	0.732 051	-0.267 949
0.414 214	0.414 214	-1	1
-1	1	0.	-1
-0.767 327	-0.767 327	1.164 525	0.164 525
1	-1	-1	1
1.303 225	1.303 225	-0.628 627	-1.628 627
-1	1	2	1
-2.414 214	-2.414 214	-1	1
1	-1	-2.732 051	-3.732 051
7.595 755	7.595 755	7.078 117	6.078 117

$k = 9$ /

$2n = 24$ (contd).

$k = 9$	$k = 10$
19	21
4.663 903	2.931 852
3.732 051	-2.732 051
2.414 214	2.414 214
-1	-2
-0.214 413	1.517 638
1	-1
-1.249 689	0.482 362
1	0
-0.414 214	-0.414 214
-0.267 949	0.732 051
0.800 199	-0.931 852
-1	1
0.800 199	-0.931 852
-0.267 949	0.732 051
-0.414 214	-0.414 214
1	0
-1.249 689	0.482 362
1	-1
-0.214 413	1.517 638
-1	-2
2.414 214	2.414 214
-3.732 051	-2.732 051
4.663 903	2.931 852

Matrix divided by 48.

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